Modern quantum many-body physics – Interacting bosons

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https://canvas.mit.edu/courses/11339

The first step to build a theory: how to label states?

One particle states

- How to label states of one boson in 1D space? $\to |x\rangle$. The most general state $|\psi\rangle = \int \mathrm{d}x \psi(x)|x\rangle$
- Energy eigenstates (momentum eigenstates) $|k\rangle = \int \mathrm{d}x \mathrm{e}^{\mathrm{i}kx}|x\rangle$, where wave vector $k = \text{int.} \times \frac{2\pi}{L}$. (The space is a 1D ring of size L)

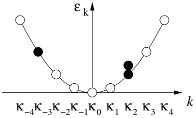
- Momentum $= p = \hbar k$.
- Energy $= \epsilon_k = \frac{\hbar^2 k^2}{2M}$ (Or $\epsilon_k = \hbar |k| c$ for massless photons)

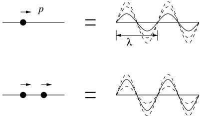
Many-particle states

- Label all zero-, one-, two-, three-, ... boson states:
  $|\emptyset\rangle$
  $|k_1\rangle$
  $|k_1, k_2\rangle$, $k_1 \leq k_2$ ($|k_1, k_2\rangle = |k_2, k_1\rangle$ for identical particles)
  $|k_1, k_2, k_3\rangle$, $k_1 \leq k_2 \leq k_3$
  ... ...
- Label all zero-, one-, two-, three-, ... boson states
  (The **second quantization** – quantum field theory of bosons):
  $n_k \equiv$ the number of bosons with wave vector k.
  $|\{n_k = 0\}\rangle$ is the ground state. $|\{n_k \neq 0\}\rangle$ is an excited state.
  $|\{n_k = 0\}\rangle = |\emptyset\rangle$. No boson
  $|\{n_{k_1} = 1, \text{others } = 0\}\rangle = |k_1\rangle$. One boson
  $|\{n_{k_1} = 1, n_{k_2} = 1, \text{others } = 0\}\rangle = |k_1, k_2\rangle = |k_2, k_1\rangle$.
  $|\{n_{k_1} = 1, n_{k_2} = 1, n_{k_3} = 1, \text{others } = 0\}\rangle = |k_1, k_2, k_3\rangle = |k_2, k_3, k_1\rangle = \cdots$.
  $|\{n_{k_1} = 2, n_{k_2} = 1, \text{others } = 0\}\rangle = |k_1, k_1, k_2\rangle = |k_1, k_2, k_1\rangle = \cdots$.

A many-boson system with no interaction $=$ a collection of decoupled harmonic oscillators

$n_k \to$ the occupation number of the bosons on orbital-k.





(a) (b)

- If we ignore the interaction between bosons $|\{n_k\}\rangle$ is an energy eigenstate with energy $E = \sum_k n_k \epsilon_k$
- The above energy can be viewed as the total energy of a collection of decoupled harmonic oscillators. The oscillators are labeled by $k = \text{int.} \times \frac{2\pi}{L}$. The oscillator labeled by k has a frequency $\omega_k = \epsilon_k/\hbar$.
- A collection of decoupled harmonic oscillators $=$ vibration modes of a vibrating string. The two polarizations of bosons $\to$ two directions of string vibrations
  $\to$ **quantum field theory** of 1D boson gas.

Many-body Hamiltonian for non-interacting bosons

View 1D non-interacting bosons (with $0, 1, 2, 3, \cdots$ bosons) as a collection of oscillators with frequencies ω_k:

$$\hat{H} = \sum_k (\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2})\hbar\omega_k, \quad \hbar\omega_k = \epsilon_k = \frac{\hbar^2 k^2}{2m}, \quad k = \text{int.} \times \frac{2\pi}{L}$$

raising-lowering operator

$$\hat{a}_k = \sqrt{\frac{m\omega_k}{2\hbar}}(\hat{x}_k + \frac{\mathrm{i}}{m\omega_k}\hat{p}_k), \qquad [\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{k,k'}$$

$$\hat{a}_k^\dagger \hat{a}_k |n_k\rangle = n_k |n_k\rangle, \qquad \hat{a}_k^\dagger |n_k\rangle = |n_k + 1\rangle, \quad \hat{a}_k |n_k\rangle = |n_k - 1\rangle.$$

- All the energy eigenstates are labeled by $|\{n_k\}\rangle = \bigotimes_k |n_k\rangle$.
  The total energy $E_{\text{tot}} = \sum_k (n_k + \frac{1}{2})\epsilon_k$.
  The total particle number $N = \sum_k n_k$.

$\hat{a}_k^\dagger, \hat{a}_k$ **are also creation-annihilation operator of bosons.**

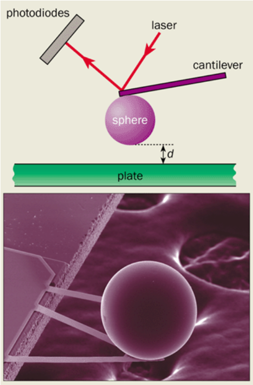
Many-body Hamiltonian for bosons on lattice

- Infinite problem on quantum field theory:
  The vaccum energy $E_0 = 0$ or $E_0 = \sum_k \frac{1}{2}\epsilon_k$?
  The right answer $E_0 = \sum_k \frac{1}{2}\epsilon_k = \infty$
- Non-interacting bosons on a lattice

  For 1D non-interacting bosons
  (with $0, 1, 2, 3, \cdots$ bosons)

  $$\hat{H} = \sum_{k\in BZ} (\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2})\epsilon_k, \quad \epsilon_k = 2t[1-\cos(ka)],$$

  $$k = \text{int.} \times \frac{2\pi}{L} \in [-\frac{\pi}{a}, \frac{\pi}{a}].$$



- The vacuum energy now is finite

$$E_0 = \sum_{k\in BZ} \frac{1}{2}\epsilon_k = L\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{\mathrm{d}k}{2\pi} 2t[1-\cos(ka)] = L\frac{2t}{a} = 2tN.$$

- The vacuum energy can be measured via **Casimir effect**.

Many-body Hamiltonian for interacting bosons on lattice

- The total particle number operator

$$\hat{N} = \sum_{k \in BZ} \hat{a}_k^\dagger \hat{a}_k = \sum_i \hat{\varphi}_i^\dagger \hat{\varphi}_i, \qquad [\hat{\varphi}_i, \hat{\varphi}_j^\dagger] = \delta_{ij}.$$

$$\hat{a}_k = \sum_{x_i} N^{-1/2} \mathrm{e}^{\mathrm{i}\, k x_i} \hat{\varphi}_i, \quad x_i = a\ i, \quad i = 1, \cdots, N;$$

- $\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$ is the number operator for bosons on orbital k.
- $\hat{n}_i = \hat{\varphi}_i^\dagger \hat{\varphi}_i$ is the number operator for bosons on site i. $\hat{\varphi}_i^\dagger, \hat{\varphi}_i$ are creation-annihilation operator of bosons at site-i.

- Many-body Hamiltonian for interacting bosons

$$H = \sum_k (\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2})\epsilon_k - \sum_i \mu \hat{n}_i + \sum_{i \leq j} V_{ij} \hat{n}_i \hat{n}_j$$

$$= \sum_k \frac{1}{2}(\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger)\epsilon_k - \sum_i \mu \hat{\varphi}_i^\dagger \hat{\varphi}_i + \sum_{i \leq j} V_{ij} \hat{\varphi}_i^\dagger \hat{\varphi}_i \hat{\varphi}_j^\dagger \hat{\varphi}_j$$

$$= \sum_i \Big[t(\hat{\varphi}_i^\dagger \hat{\varphi}_i + \hat{\varphi}_i \hat{\varphi}_i^\dagger) - t(\hat{\varphi}_{i+1}^\dagger \hat{\varphi}_i + \hat{\varphi}_i^\dagger \hat{\varphi}_{i+1}) \Big] - \sum_i \mu \hat{\varphi}_i^\dagger \varphi_i + \sum_{i \leq j} V_{ij} \hat{\varphi}_i^\dagger \hat{\varphi}_i \hat{\varphi}_j^\dagger \hat{\varphi}_j$$

Hard-core bosons and spin-1/2 systems

- Assume on-site interaction $V_{ij} = U\delta_{ij}, \quad \mu = U + 2B + t \to$
  $U\hat{n}_i\hat{n}_i - \mu\hat{n}_i = U(\hat{n}_i - 1)\hat{n}_i - (2B + t)\hat{n}_i, \quad U \to +\infty$
  The low energy sector for interaction $\to$ $n_i = 0, 1$ $(\downarrow, \uparrow)$ or
  $$n_i = \frac{\sigma_i^z - 1}{2}, \quad \hat{\varphi}_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sigma_i^- = \frac{\sigma_i^x - \mathrm{i}\sigma_i^y}{2}.$$
  Hamiltonian for interacting bosons = a spin-1/2 system
  $$H_{\text{XY-model}} = \sum_i \Big[- t(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+) - B\sigma_i^z \Big]$$
  $$= \sum_i \Big[- J(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y) - B\sigma_i^z \Big], \qquad J = \frac{1}{2}t$$
- $U(1)$ symmetry generated by $U_\phi = \prod_i \mathrm{e}^{\mathrm{i}\phi\sigma_i^z/2}$: $U_\phi H U_\phi = H$.
  $\sum_i \sigma_i^z \sim N + const.$ conservation.
- Phase diagram: Treat operators $\boldsymbol{\sigma}$ as classical unit-vector (spin) $\boldsymbol{n}$.

$B < 0 : |\downarrow \cdots \downarrow\rangle \qquad B \sim 0 : |\to \cdots \to\rangle \qquad B > 0 : |\uparrow \cdots \uparrow\rangle$

$\longrightarrow B$

0-boson/site   Superfluid   1-boson/site (Mott insulator)

- Assume on-site interaction to have a form $U[(n_i-1)^4-(n_i-1)^2]$. The low energy sector for the interaction: $n_i = 0,1,2$ $(\downarrow, 0, \uparrow)$ or

$$n_i = S_i^z - 1, \quad \hat{\varphi}_i = S_i^- .$$

Hamiltonian for interacting bosons = a spin-1 system ($U(1)$ symm.)

$$H = \sum_i \Big[-t(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) - BS_i^z + V(S_i^z)^2 \Big]$$

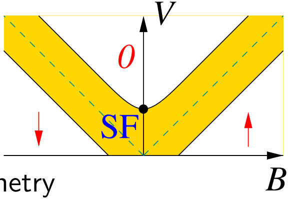
$$= \sum_i \Big[-J(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) - BS_i^z + V(S_i^z)^2 \Big].$$

- B-V phase diagrame Treat operators $\boldsymbol{\sigma}$ as classical unit-vector (spin) $\boldsymbol{n}$.

- Two different critical points:

- The black-line represents a $z=2$ critical point. (*ie* excitations have dispertion relation $\omega_k \sim k^2$)

- The filled dot represents a different $z=1$ critical point with emergent Lorentz symmetry (*ie* excitations have dispertion relation $\omega_k \sim k$)



$J = 1$

Many-body Hamiltonian

- Consider a system formed by two spin-1/2 spins. The spin-spin interaction: $H = J(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y + \sigma_1^z\sigma_2^z)$.
  where $\sigma_i^{x,y,z}$ are the Pauli matrices acting on the i^{th} spin.
  $J < 0 \to$ ferromagnetic, $J > 0 \to$ antiferromagnetic.
  Is H a two-by-two matrix? In fact
  $H = -J[(\sigma^x \otimes I)\cdot(I\otimes\sigma^x) + (\sigma^y\otimes I)\cdot(I\otimes\sigma^y) + (\sigma^z\otimes I)\cdot(I\otimes\sigma^z)]$
  H is a four-by-four matrix:

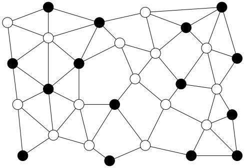
$$\sigma_1^z\sigma_2^z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \sigma_1^x\sigma_2^x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \sigma_1^x\sigma_2^z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

- $\sigma_i^z = I\otimes\cdots\otimes I\otimes\sigma^z\otimes I\otimes\cdots\otimes I$ is a $2^{N_{\text{site}}}$-dimensional matrix
  Example: An 1D ring formed by L spin-1/2 spins:

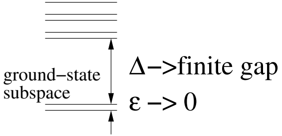
$$H = -\sum_{i=1}^{L}\sigma_i^x\sigma_{i+1}^x - h\sum_{i=1}^{L}\sigma_i^z$$

– transverse Ising model. H is a $2^L \times 2^L$ matrix.

Condensed matter: A local many-body quantum system



- A many-body quantum system
  $=$ Hilbert space $\mathcal{V}_{tot}$ + Hamiltonian H
  - The locality of the Hilbert space:
    $\mathcal{V}_{tot} = \otimes_{i=1}^{N} \mathcal{V}_i$
  - The i also label the vertices of a graph
- A local Hamiltonian $H = \sum_{\boldsymbol{x}} H_{\boldsymbol{x}}$ and $H_{\boldsymbol{x}}$ are local hermitian operators acting on a few neighboring $\mathcal{V}_i$'s.
- A quantum state, a vector in $\mathcal{V}_{tot}$:
  $|\Psi\rangle = \sum \Psi(m_1, ..., m_N)|m_1\rangle \otimes ... \otimes |m_N\rangle$,
  $|m_i\rangle \in \mathcal{V}_i$
- A gapped Hamiltonian has the following spectrum as $N \to \infty$
  (eg $H = -\sum(J\sigma_i^z\sigma_{i+\delta}^z + h\sigma_i^x)$)



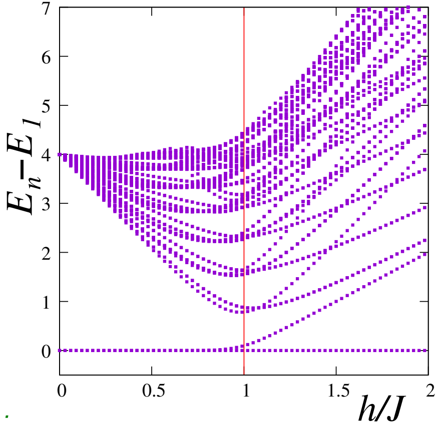
Many-body spectrum using Octave (Matlab or Julia)

Transverse Ising model on a ring of L site:

$$H = -J \sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x - h \sum_{i=1}^{L} \sigma_i^z$$

H is an 2^L-by-2^L matrix, whose eigenvalues can be computed via the following Octave code
(the code also run in Matlab or Julia with minor modifications):

```
X=sparse([0,1;1,0]); Z=sparse([1,0;0,-1]); XX=kron(X,X);
L=13; h=1.0; J=1.0
H=-kron(kron(X, speye(2^(L-2)) ),X);
for i=1:L-1
  H=H - kron( speye(2^(i-1)), kron(J*XX, speye(2^(L-1-i)))) ;
end
for i=1:L
  H=H - kron( speye(2^(i-1)), kron(h*Z, speye(2^(L-i)))) ;
end
eigs( H , 10, 'sa') # compute the lowest 10 eigenvalues
```

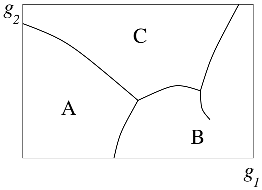


The 100 lowest energy eigenvalues for $L = 16$, as functions of $h/J \in [0, 2]$.

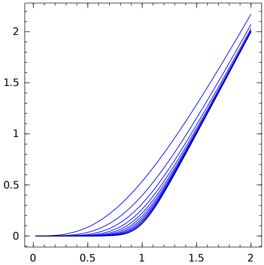
Quantum phases and quantum phase transitions

- Phases are defined through phase transitions.
  What are phase transitions?

  As we change a parameter g in Hamiltonian $H(g)$, the ground state energy density $\epsilon_g = E_g/V$ or the average of a local operator $\langle \hat{O} \rangle$ may have a singularity at g_c: the system has a phase transition at g_c.
  The Hamiltonian $H(g)$ is a smooth function of g. How can the ground state energy density ϵ_g be singular at a certain g_c?



$E_2 - E_1$ of trans. Ising for $L = 3, \cdots, 13$



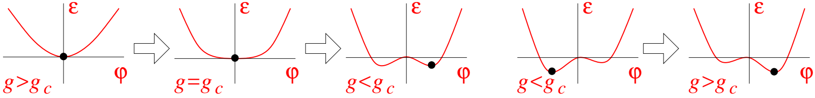
- There is no singularity for finite systems.
  Singularity appears only for infinite systems.
- Spontaneous symmetry breaking is a mechanism to cause a singularity in ground state energy density ϵ_g.
  $\to$ Spontaneous symmetry breaking causes phase transition.

Symmetry breaking theory of phase transition

It is easier to see a phase transition in the semi classical approximation of a quantum theory.

- Variational ground state $|\Psi_\phi\rangle$ for H_g is obtained by minimizing energy $\epsilon_g(\phi) = \frac{\langle\Psi_\phi|H_g|\Psi_\phi\rangle}{V}$ against the variational parameter ϕ.
  $\epsilon_g(\phi)$ is a smooth function of ϕ and g. How can its minimal value $\epsilon_g \equiv \epsilon_g(\phi_{min})$ have singularity as a function of g?
- Minimum splitting → singularity in $\frac{\partial^2 \epsilon_g}{\partial g^2}$ at g_c. Second order trans.
  State-B has less symmetry than state-A.
  State-A → State-B: spontaneous symmetry breaking.

\- For a long time, we believe that
phase transition = change of symmetry
the different phases = different symmetry.



- Minimum switching → singularity in $\frac{\partial \epsilon_g}{\partial g}$ at g_c. First order trans.

Example: meanfield symmetry breaking transition

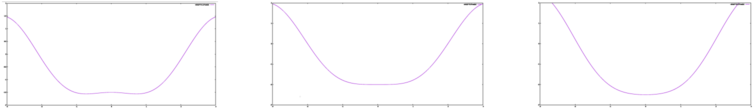
Consider a transverse field Ising model $H = \sum_i -J\sigma_i^x\sigma_{i+1}^x - h\sigma_i^z$

Use trial wave function $|\Psi_\phi\rangle = \otimes_i|\psi_i\rangle$, $|\psi_i\rangle = \cos\frac{\phi}{2}|\uparrow\rangle + \sin\frac{\phi}{2}|\downarrow\rangle$ to estimate the ground state energy

$\langle\Psi_\phi|H|\Psi_\phi\rangle = -\sum\langle\psi_i|\sigma_i^x|\psi_i\rangle\langle\psi_{i+1}|\sigma_{i+1}^x|\psi_{i+1}\rangle - h\sum\langle\psi_i|\sigma_i^z|\psi_i\rangle.$

$= (2J\cos\frac{\phi}{2}\sin\frac{\phi}{2})^2 - h(\cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}) = \sin^2\phi - h\cos\phi$

Phase transition at $h/J = 2$. ($h/J = 1.5, 2.0, 2.5$)



Order parameter and symmetry-breaking phase transition

ϕ or σ_i^x are order parameters for the Z_2 symm.-breaking transition:

- Under Z_2 (180° S^z rotation), $\phi \to -\phi$ or $\sigma_i^x \to -\sigma_i^x$
- In symmetry breaking phase $\phi = \pm\phi_0$, $\langle\sigma_i^x\rangle = \pm$.

In symmetric phase $\phi = 0$, $\langle\sigma_i^x\rangle = 0$. (**Classical picture**)

Ginzberg-Landau theory of continuous phase transition

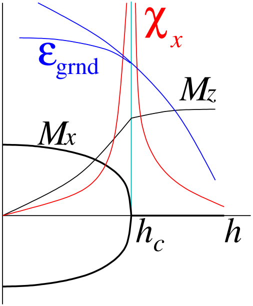
- Quantum Z_2-Symmetry: generator $U = \prod_j \sigma^z_j$, $U^2 = 1$.
  Symmetry trans.: $U\sigma^z_i U^\dagger = \sigma^z_i$, $U\sigma^x_i U^\dagger = -\sigma^x_i$, $U\sigma^y_i U^\dagger = -\sigma^y_i$.
  $\to UHU^\dagger = H$. If $H|\psi\rangle = E_{\text{grnd}}|\psi\rangle$, then $UH|\psi\rangle = E_{\text{grnd}} U|\psi\rangle \to$
  $UHU^\dagger U|\psi\rangle = E_{\text{grnd}} U|\psi\rangle \to HU|\psi\rangle = E_{\text{grnd}} U|\psi\rangle$
  Both $|\psi\rangle$ and $U|\psi\rangle$ are ground states of H:
  Either $|\psi\rangle \propto U|\psi\rangle$ (symmetric) or $|\psi\rangle \not\propto U|\psi\rangle$ (symm.-breaking).
- Trial wave function $|\Psi_\phi\rangle = \bigotimes_i(\cos\frac{\phi}{2}|\uparrow\rangle_i + \sin\frac{\phi}{2}|\downarrow\rangle_i)$: $U|\Psi_\phi\rangle = |\Psi_{-\phi}\rangle$
  $\to \langle\Psi_\phi|H|\Psi_\phi\rangle = \langle\Psi_\phi|U^\dagger UHU^\dagger U|\Psi_\phi\rangle = \langle\Psi_{-\phi}|H|\Psi_{-\phi}\rangle \to$
  $\epsilon(h,\phi) = \epsilon(h,-\phi)$
- If $|\Psi_{\phi=0}\rangle$ is the ground state $\to$ symmetric phase.
  If $|\Psi_{\phi\neq 0}\rangle$ is the ground state $\to$ symmetry breaking phase.
- Near the phase transition ϕ is small $\to$

$$\epsilon(h,\phi) = \epsilon_0(h) + \frac{1}{2}a(h)\phi^2 + \frac{1}{4}b(h)\phi^4 + \cdots$$

  Transition happen at $a(h_c) = 0$.

Properties near the $T = 0$ (quantum) phase transition

- Ground state energy density:
  $\phi = 0,\ \epsilon_{\text{grnd}}(h) = \epsilon_0(h)$ if $a(h) > 0$
  $\phi = \pm\sqrt{\frac{-a}{b}},\ \epsilon_{\text{grnd}}(h) = \epsilon_0(h) - \frac{1}{4}\frac{a(h)^2}{b}$ if $a(h) < 0$
  $\epsilon_{\text{grnd}}(h)$ is non-analytic at the transition point: $a(h) = a_0(h - h_c)$:
  $$\epsilon_{\text{grnd}}(h) = \begin{cases} \epsilon_0(h), & h > h_c \\ \epsilon_{\text{grnd}}(h) = \epsilon_0(h) - \frac{1}{4}\frac{a_0(h-h_c)^2}{b}, & h < h_c \end{cases}$$
- Magnetization in z-direction: $M_z = \frac{\partial \epsilon_{\text{grnd}}(h)}{\partial h}$.
  $M_z = \frac{\partial \epsilon_0(h)}{\partial h}, \quad h > h_c$
  $M_z = \frac{\partial \epsilon_0(h)}{\partial h} - \frac{1}{2}\frac{a_0(h-h_c)}{b}, \quad h < h_c$
  $\to \Delta M_z \sim |\Delta h|$
- Magnetization in x-dir.: $M_x = \langle \sigma^x \rangle = \sin\phi$
  $\phi = \pm\sqrt{\frac{-a(h)}{b}} \to \Delta M_x \sim |\Delta h|^{1/2}$
- Magnetic susceptibility in x-direction:
  From $\epsilon(h, \phi, h_x) = \frac{1}{2}a(h)\phi^2 - h_x\phi + \cdots$
  $\to M_x = \phi = \frac{1}{a(h)} \to \chi_x = \frac{1}{a(h)} \to \Delta\chi_x \sim |\Delta h|^{-1}$



Quantum picture of continuous phase transition

No symmetry breaking in quantum theory according: If $[H, U] = 0$, then H and U share a commom set of eigenstates. The ground state $|\Psi_{\text{grnd}}\rangle$ of H, is an eigenstate of U: $U|\Psi_{\text{grnd}}\rangle = e^{i\theta}|\Psi_{\text{grnd}}\rangle$.
No symmetry breaking.
$|\Psi_\phi\rangle$ and $|\Psi_{-\phi}\rangle$ in semi classical approximation are not true ground states. The true ground state is $|\Psi_{\text{grnd}}\rangle = |\Psi_\phi\rangle + |\Psi_{-\phi}\rangle$ which do not break the symmetry.

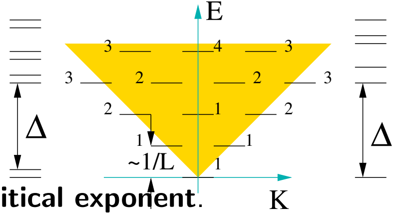
- **Quantum picture**: Symmetry-breaking order parameter is zero, $\langle\Psi_{\text{grnd}}|\sigma_i^x|\Psi_{\text{grnd}}\rangle = 0$, for the true ground state. But **the ground states, $|\Psi_{\text{grnd}}\rangle = |\Psi_\phi\rangle + |\Psi_{-\phi}\rangle$ and $|\Psi'_{\text{grnd}}\rangle = |\Psi_\phi\rangle - |\Psi_{-\phi}\rangle$, have an exponentially small energy separation $\Delta \sim e^{-L/\xi}$.**
Symmetry-breaking order parameter is non-zero only for approximate ground states, $|\Psi_\phi\rangle$ and $|\Psi_{-\phi}\rangle$.
- **Detect symmetry breaking from correlation function**: $\lim_{|i-j|\to\infty}\langle\Psi_{\text{grnd}}|\sigma_i^x\sigma_j^x|\Psi_{\text{grnd}}\rangle = const.$.
Symmetric phase: $\lim_{|i-j|\to\infty}\langle\Psi_{\text{grnd}}|\sigma_i^x\sigma_j^x|\Psi_{\text{grnd}}\rangle = 0$

- From the energy $\epsilon(h,\phi) = \epsilon_0(h) + \frac{1}{2}a(h)\phi^2 + \frac{1}{4}b(h)\phi^4 + \cdots$
  $\to$ Restoring force $f = -a\phi - b\phi^3$ $\to$ EOM $\rho\ddot{\phi} = -a\phi - b\phi^3$.
- $k \neq 0$ mode: $\epsilon(h,\phi) = \frac{1}{2}g(\partial_x\phi)^2 + \frac{1}{2}a(h)\phi^2 + \frac{1}{4}b(h)\phi^4 + \cdots$
  Restoring force $f = g\partial_x^2\phi - a\phi - b\phi^3$
  $\to$ EOM $\rho\ddot{\phi} = g\partial_x^2\phi - a\phi - b\phi^3$.

Where does ρ come from?

- Collective mode: $\omega_k = \sqrt{\frac{gk^2+a}{\rho}}$

  Energy gap: $\Delta = \sqrt{\frac{a(h)}{\rho}} = \sqrt{\frac{a_0(h-h_c)}{\rho}}$.



\- At the critical point $h = h_c$:
Gapless = diverging susceptibility
$\omega_k \sim k^z$, $z = 1$. z is the **dynamical critical exponent**.
$z = 1 \to$ Emergence of Lorentz symmetry.

Continuous quantum phase transition between gapped phases = gap closing phase transition. Continuous quantum phase transition between gapless phases : more low energy modes at the critical point.

Collective mode of order parameter ϕ: calculate

Consider a transverse field Ising model $H = -\sum_i (J\sigma_i^x \sigma_{i+1}^x + h\sigma_i^z)$.
Trial wave function $|\Psi_{\phi_i}\rangle = \otimes_i |\phi_i\rangle$, $|\phi_i\rangle = \frac{|\uparrow\rangle + \phi_i|\downarrow\rangle}{\sqrt{1+|\phi_i|^2}}$ (Key: ϕ_i complex)

$$\langle \sigma_i^x \rangle = \frac{\phi_i + \phi_i^*}{1+|\phi_i|^2}, \quad \langle \sigma_i^z \rangle = \frac{1-|\phi_i|^2}{1+|\phi_i|^2}.$$

- Average energy

$$\bar{H} = -\sum_i \Big[J \frac{(\phi_i+\phi_i^*)(\phi_{i+1}+\phi_{i+1}^*)}{(1+|\phi_i|^2)(1+|\phi_{i+1}|^2)} + h \frac{1-|\phi_i|^2}{1+|\phi_i|^2} \Big]$$

Geometric phase term

$$\langle \phi_i | \frac{\mathrm{d}}{\mathrm{d}t} |\phi_i\rangle = \frac{\phi_i^* \dot{\phi}_i}{1+|\phi_i|^2} + (1+|\phi_i|^2)^{1/2} \frac{\mathrm{d}}{\mathrm{d}t} (1+|\phi_i|^2)^{-1/2}$$

$$= \frac{\phi_i^* \dot{\phi}_i}{1+|\phi_i|^2} - \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \log(1+|\phi_i|^2)$$

Phase space Lagrangian (quadratic approximation: $\phi_i = q_i + \frac{\mathrm{i}}{2} p_i$ small)

$$L = \langle \Phi_{\phi_i} | \mathrm{i}\,\tfrac{\mathrm{d}}{\mathrm{d}t} - H | \Phi_{\phi_i} \rangle = \sum_i \mathrm{i}\phi_i^* \dot{\phi}_i + J(\phi_i + \phi_i^*)(\phi_{i+1} + \phi_{i+1}^*) - 2h|\phi_i|^2$$

$$= \sum_i \big[p_i \dot{q}_i + 4J q_i q_{i+1} - 2h(q_i^2 + \tfrac{1}{4} p_i^2) \big]$$

Collective mode of order parameter ϕ: calculate

EOM:

$$\dot{q}_i = \frac{\partial \bar{H}}{\partial p_i} = \frac{h}{2} p_i, , \qquad \dot{p}_i = -\frac{\partial \bar{H}}{\partial q_i} = 4J(q_{i+1} + q_{i-1}) - 4hq_i$$

in k-space ($q_i = \sum_k N^{-1/2} \mathrm{e}^{\mathrm{i}\, kia} q_k,\ p_i = \sum_k N^{-1/2} \mathrm{e}^{\mathrm{i}\, kia} p_k$):

$$\dot{q}_k = \frac{h}{2} p_k, , \qquad \dot{p}_k = 4(J\mathrm{e}^{\mathrm{i}\, ka} + J\mathrm{e}^{-\mathrm{i}\, ka} - h) q_k$$

k label harmonic oscillators with EOM

$$\ddot{q}_k = 2h[2\cos(ka) - h] q_k \quad \to \quad -\omega_k^2 = 2h[2J\cos(ka) - h]$$

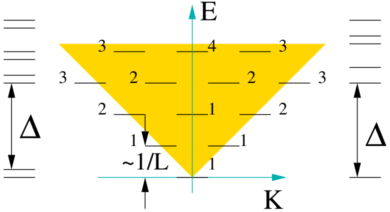
The dispersion of the collective mode

$$\omega_k = \sqrt{2h[h - 2J\cos(ka)]}$$

- For $h > 2J$, gap $= \sqrt{2h(h - 2J)}$.
  For $h = 2J$, gapless mode with velocity $v = 2aJ$ and $\omega_k = v|k|$.

Many-body spectrum at the critical point

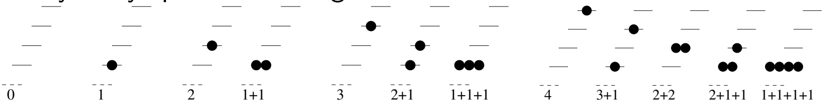
- At the critical point, the gapless excitation is described by a real scaler field ϕ (or q_i) with EOM:



$$\ddot{\phi} = v^2 \partial_x^2 \phi.$$

$=$ an oscillator for every $k = \frac{2\pi}{L} n$
$=$ a wave mode with $\omega_k = v|k|$
$=$ a boson with $\epsilon(p) = v|p|$

- Many-body spectrum for right movers:



1 1 2 3 5

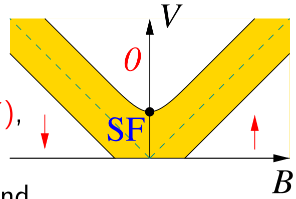
Do not count for the $k = 0$ orbital.

- Total energy and total momentum for right movers $E = vK$.
Magic at critical point: Emergence of Lorentz invariance $\epsilon = vk$.
Emergence of independent right-moving and left-moving sectors (extra degeneracy in mony-body spectrum): conformal invariance

$z = 1$ and $z = 2$ critical points

The transverse Ising model, $H = -\sum_i (J\sigma_i^x \sigma_{i+1}^x + h\sigma_i^z)$,

has $z = 1$ critical points at $h = \pm J$



The spin-1 XY model,
$H = \sum_i (-JS_i^x S_{i+1}^x - JS_i^y S_{i+1}^y + V(S_i^z)^2 - BS_i^z)$,
has $z = 1$ and $z = 2$ critical points.

- The $z = 1$ criticial point appears when $B = 0$ and the spin-1 XY model has the $S^z \to -S^z$ symmetry.

- The phase space Lagrangian of has a form $\mathcal{L} = A\phi^* \dot{\phi} + B\dot{\phi}^* \dot{\phi} - C|\partial\phi|^2$ for the collective mode at the criticial point. When $B = 0$, $A = 0$, which leads to the $z = 1$ critical point. When $B \neq 0$, $A \neq 0$, which leads to the $z = 2$ critical point.

The minimal value of dynamical exponent z is 1

- The $z = 2$ critical point can appear if we have $U(1)$ spin rotation symmetry in the S^x-S^y plane. In this case, the critical point describe the transition from a gapped Mott insulator (spin polarized) phase to a gapless superfluid (XY spin order) phase ($U(1)$ symmetry breaking phase) with $z = 1$ (*ie* $\omega \sim k$).

\- The gapless is the Goldstone mode. **Spontaneous breaking of a continuous symmetry always give rise to a gapless model.**

\- The critical point always has more low energy excitations then the two phases it connects.

- The $z = 1$ critical point can appear if we have Z_2 spin rotation symmetry in the $S^x \to -S^x$. In this case, the critical point describe the transition from a gapped symmetric phase to a gapped spontaneous Z_2-symmetry breaking phase.
- $z < 1, \ \omega \sim |k|^z$ **is not allowed for short range interaction**, since the velocity for any excitations has an upper bound $v \lesssim a||H_{i,i+a}||/\hbar$

The property of $\boldsymbol{k} = 0$ mode (quadratic approx. valid?)

- Now consider transverse Ising model in d dimensions ($g \sim J, h$)

$$L = \sum_{\boldsymbol{i}} \sum_{\mu = \boldsymbol{x}, \boldsymbol{y}, \cdots} \left[p_{\boldsymbol{i}} \dot{q}_{\boldsymbol{i}} + 4J q_{\boldsymbol{i}} q_{\boldsymbol{i}+\mu} \right] - \sum_{\boldsymbol{i}} \left[2h(q_{\boldsymbol{i}}^2 + \frac{1}{4} p_{\boldsymbol{i}}^2) - g q_{\boldsymbol{i}}^4 \right]$$

  The transition point now is at $h = 2dJ$

- At the critical point $h = 2dJ$,
  the $\boldsymbol{k} = 0$ mode is described by the Lagrangian

$$L = N p \dot{q} - \frac{N}{2} h p^2 - N g q^4$$
$$= \tilde{p} \dot{\tilde{q}} - \frac{h}{2} \tilde{p}^2 - \frac{g}{N} \tilde{q}^4, \qquad \tilde{p} = \sqrt{N} p, \quad \tilde{q} = \sqrt{N} q.$$

- The zero-point energy from the $\boldsymbol{k} = 0$ mode $\tilde{p}\tilde{q} \sim 1 \to \tilde{q} \sim N^{1/6}$

$$\text{mininizing:} \quad \frac{h}{2} \tilde{p}^2 + \frac{g}{N} \tilde{q}^4 \sim \frac{h}{2} \tilde{q}^{-2} + \frac{g}{N} \tilde{q}^4 \sim J N^{-1/3}$$

  The non-linear term is important for $\boldsymbol{k} = 0$ mode.

- The zero-point energy from the $\boldsymbol{k}$ mode (ignoring the non-linear term)
  $Jk \sim J N^{-1/d} |_{k \sim N^{-1/d}}$

The non-linear effect for $\boldsymbol{k}$ mode

- At the critical point $h = 2dJ$, the $\boldsymbol{k}$ mode is described by the Lagrangian

$$\begin{aligned} L &= Np\dot{q} - JN\boldsymbol{k}^2 q^2 - \frac{N}{2} h p^2 - Ngq^4 \\ &= \tilde{p}\dot{\tilde{q}} - J\boldsymbol{k}^2 \tilde{q}^2 - \frac{h}{2}\tilde{p}^2 - \frac{g}{N}\tilde{q}^4, \qquad \tilde{p} = \sqrt{N}p, \quad \tilde{q} = \sqrt{N}q. \end{aligned}$$

- The zero-point energy from the $\boldsymbol{k}$ mode $\tilde{p}\tilde{q} \sim 1 \to \tilde{p} \sim 1/\tilde{q} \sim \sqrt{k}$

$$J\boldsymbol{k}^2\tilde{q}^2 + \frac{h}{2}\tilde{p}^2 + \frac{g}{N}\tilde{q}^4 \sim Jk + \frac{h}{2}k + \frac{g}{Nk^2}$$

The non-linear term is important if

$$\frac{g}{Nk^2} > Jk \quad \text{or} \quad k < \frac{1}{N^{1/3}}$$

- Since the smallest k is $\frac{1}{N^{1/d}}$. For $d > 3$ there is no $\boldsymbol{k}$ satisfying the above condition (except $\boldsymbol{k} = 0$). We can ignore the non-linear term. **Our critical theory from quadratic approximation is correct**.
- For $d \leq 3$, we cannot ignore the non-linear term. **Our critical theory from quadratic approximation is incorrect**.

Quantum fluctuations: relevant/irrelevant perturbations

EOM of Z_2 order parameter for the $d+1$D-transverse Ising model

$$\rho\ddot{\phi} = g\partial_{\boldsymbol{x}}^2\phi + a\phi + b\phi^3$$

Is the $b\phi^3$ term importent at the transition point $a = 0$?

- The action $S = \int \mathrm{d}t\,\mathrm{d}^d\boldsymbol{x}\ [\frac{1}{2}\rho(\dot{\phi})^2 - \frac{1}{2}g(\partial_{\boldsymbol{x}}\phi)^2 - \frac{1}{2}a\phi^2 - \frac{1}{4}b\phi^4]$

- Treating the above as a quantum system with quatum fluctuations, the term $\frac{1}{4}b\phi^4$ is irrelevant if dropping it does not affect the low energy properties at critical point $a = 0$. Otherwise, it is revelvent.
- Rescale t to make $\rho = g$ and rescale ϕ to make $\rho = g = 1$.

- Consider the fluctuation at length scale ξ. The action for such fluctuation is $S_\xi = \int \mathrm{d}t\ [\frac{1}{2}\xi^d(\dot{\phi})^2 - \frac{1}{2}\xi^{d-2}\phi^2 - \frac{1}{4}b\xi^d\phi^4]$

  $\to$ Oscillator with mass $M = \xi^d$ and spring constant $K = \xi^{d-2}$.

  Oscillator frequency $\omega = \sqrt{K/M} = 1/\xi$.

  Potential energy for quantum fluctuation $E = \frac{1}{2}\omega = \frac{1}{2}\xi^{d-2}\phi^2$.

  Fluctuation $\phi^2 = \xi^{1-d}$.

Compare $\xi^{d-2}\phi^2$ and $b\xi^d\phi^4$: $\frac{b\xi^d\phi^4}{\xi^{d-2}\phi^2} = b\xi^{3-d}$ for $\xi \to \infty$, we conclude **the $b\phi^4$ term is irrelevant for $d > 3$. Relevant for $d < 3$**

Simple rules to test relevant/irrelevant perturbations

- After rescaling t to make $\rho = g$ and rescaling ϕ to make $\rho = g = 1$, the action becomes $S = \int \mathrm{d}t\,\mathrm{d}^d x\, [\frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\partial_x \phi)^2 - \frac{1}{2}a\phi^2 - \frac{1}{4}b\phi^4]$
- **Estimate from dimensional analysis**:
  $[S] = [L]^0$ (from $\mathrm{e}^{-\mathrm{i}S}$). $[t] = [L]$ (from $\frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\partial_x \phi)^2$)
  $[\phi] = [L]^{\frac{1-d}{2}}$, $[a] = L^{-2}$, $[b] = [L]^{d-3}$

- **Counting dimensions**:
  $[t] = -1$, $[S] = 0$.
  $[\phi] = \frac{d-1}{2}$, $[a] = 2$, $[b] = 3 - d$.
- From the scaling dimensions, we can see that the quantum fluctuations of ϕ^2 are given by $\phi^2 \sim L^{1-d}$, and the dimensionless ratio of $L^d \frac{1}{L^2}\phi^2$ and $L^d b\phi^4$ terms is given by $\frac{bL^d\phi^4}{L^{d-2}\phi^2} \sim bL^{3-d}$
  The $b\phi^4$ term is irrelevant if $[b] < 0$. Relevant if $[b] > 0$.
  The $a\phi^2$ term is always relevant since $[a] = 2 > 0$.
- **More precise definition of scaling dimension**:
  The correlation of ϕ at the critical point $a = b = 0$
  $\langle \phi(x)\phi(y) \rangle = \frac{1}{|x-y|^{2h_\phi}}$. h_ϕ is the scaling dimension of ϕ: $h_\phi = \frac{d-1}{2}$.

Specific heat at the critical point

- Thermal energy density

$$\epsilon_T = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \frac{v|k|}{\mathrm{e}^{v|k|/k_B T} - 1} = 2\frac{k_B^2 T^2}{2\pi v} \int_0^{+\infty} \mathrm{d}x \frac{x}{\mathrm{e}^x - 1} = \frac{k_B^2 T^2}{v} \frac{\pi}{6}$$

where $\int_0^{+\infty} \mathrm{d}x \frac{x}{\mathrm{e}^x-1} = \frac{\pi^2}{6}$

- Specific heat

$$c_T = \frac{\partial \epsilon_T}{\partial T} = k_B \frac{k_B T}{v} \frac{\pi}{3} = \Big(\frac{\pi}{6} k_B \frac{k_B T}{v}\Big)_R + \Big(\frac{\pi}{6} k_B \frac{k_B T}{v}\Big)_L$$

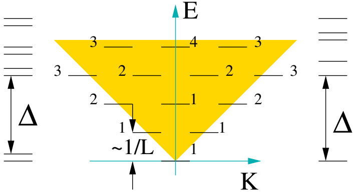
- *The above result is incorrect. The correct one is*

$$c_T = \Big(\frac{1}{2}\frac{\pi}{6} k_B \frac{k_B T}{v}\Big)_R + \Big(\frac{1}{2}\frac{\pi}{6} k_B \frac{k_B T}{v}\Big)_L$$

- $\frac{1}{2} = c$ is called the **central charge** = number of modes.
- Many-body spectrum for one right-moving mode ($c = 1$):
  $1, 1, 2, 3, 5, 7, 11, \cdots$ = partition number

Specific heat away from the critical point

Away from the critical point, the boson dispersion becomes $\epsilon_k = \sqrt{v^2k^2 + \Delta^2}$ where Δ is the many-body spectrum gap on a **ring** (the energy to create a single boson).



Many-body spectrum for a ring

many-body spectrum = spectrum of the set of the oscillators

($\times 2$ in the symmetry breaking phases)

Specific heat

$$c \sim T^\alpha \mathrm{e}^{-\frac{\Delta}{k_B T}}$$

The above result is correct in the symmetric phase, but incorrect in the symmetry breaking phase. The correct one is

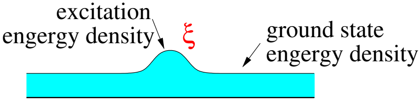
$$c \sim T^\alpha \mathrm{e}^{-\frac{\Delta/2}{k_B T}}$$

Remark: The gap in many-body spectrum for an open line is $\Delta/2$.

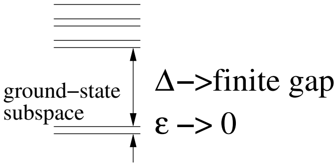
The answer is very different for gapped system and gapless systems. Here, we only consider the definition of quasiparticle for gapped systems.

Consider a many-body system $H_0 = \sum_x H_x$, with ground state $|\Psi_{\text{grnd}}\rangle$.

- a point-like excitation above the ground state is a many-body wave function $|\Psi_\xi\rangle$ that has an energy bump at location ξ: energy density $= \langle\Psi_\xi|H_{\boldsymbol{x}}|\Psi_\xi\rangle$



More precisely, point-like excitations at locations ξ_i are something that can be trapped by local traps δH_{ξ_i}: $|\Psi_{\xi_i}\rangle$ is the gapped ground state of $H_0 + \sum_i \delta H_{\xi_i}$ – the Hamiltonian with traps.



Local and topological excitations

Consider a many-body state $|\Psi_{\xi_1,\xi_2,\cdots}\rangle$ with several point-like excitations at locations ξ_i.

Can the first point-like excitation at ξ_1 be created by a local operator O_{ξ_1} from the ground state: $|\Psi_{\xi_1,\xi_2,\cdots}\rangle = O_{\xi_1}|\Psi_{\xi_2,\cdots}\rangle$?

$|\Psi_{\xi_1,\xi_2,\cdots}\rangle$ = the ground state of $H_0 + \delta H_{\xi_1} + \delta H_{\xi_1} + \cdots$

$|\Psi_{\xi_2,\cdots}\rangle$ = the ground state of $H_0 + \delta H_{\xi_1} + \cdots$

If yes: the point-like excitation at ξ_1 is a **local** excitation

If no: the point-like excitation at ξ_1 is a **topological** excitation

Local and topological excitations

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$|\Psi_{\xi_1,\xi_2,\cdots}\rangle$ = the ground state of $H_0 + \delta H_{\xi_1} + \delta H_{\xi_1} + \cdots$

$|\Psi_{\xi_2,\cdots}\rangle$ = the ground state of $H_0 + \delta H_{\xi_1} + \cdots$

If yes: the point-like excitation at ξ_1 is a **local** excitation

If no: the point-like excitation at ξ_1 is a **topological** excitation

Example: Consider an 1D Ising model $H_0 = -J\sum_i \sigma_i^z\sigma_{i+1}^z$ with

one of the degenerate ground states $|\Psi_0\rangle = |\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle$

a state w/ three point-like excitations $|\Psi_{\xi_1\xi_2\xi_3}\rangle = |\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow\rangle$

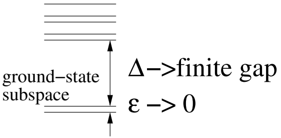
ξ_1 ξ_2 ξ_3 - The point-like excitation at ξ_1 is a spin flip created by $\sigma^x_{\xi_1}$ – a local excitation.

- The point-like excitations at ξ_2, ξ_3 are topological excitations that cannot be created by any local operators.

The pair can be created by a string operator $W_{\xi_2\xi_3} = \prod_{i=\xi_2}^{\xi_3} \sigma_i^x$.

Experimental consequence of topological excitations

- The topological topological excitations are **fractionalized** local excitations: a spin-flip can be viewed as a bound state of two wall excitations spin-flip = wall $\otimes$ wall. $|\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow\rangle$
- Energy cost of spin-flip $\Delta_{\text{flip}} = 4J$
  Energy cost of domain wall $\Delta_{\text{wall}} = 2J$.
- The many-body spectrum gap on a ring $\Delta = \Delta_{\text{flip}} = 2\Delta_{\text{wall}}$. This gap can be measured by neutron scattering.



- The thermal activation gap measured by specific heat $c \sim T^{\alpha} e^{-\frac{\Delta_{\text{therm}}}{k_B T}}$ is $\Delta_{\text{therm}} = \Delta_{\text{wall}}$.

  The difference of the neutron gap Δ and the thermal activation gap Δ_{therm} $\to$ fractionalization.

Another example: 1D spin-dimmer state

Consider a $SO(3)$ spin rotation symmetric Hamiltonian H_0 whose ground states are spin-dimmer state formed by spin-singlets, which break the translation symmetry but not spin rotation symmetry:

$$(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)$$
$$\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow$$

- Local excitation = spin-1 excitation

$$(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)\uparrow\uparrow(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)$$

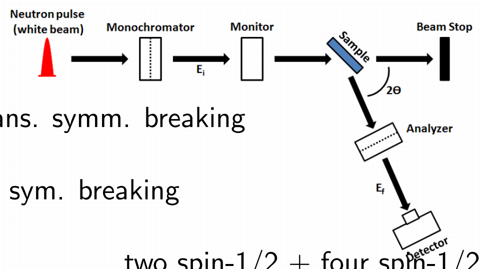
- Topo. excitation (domain wall) = spin-1/2 excitation (spinon)

$$(\uparrow\downarrow)(\uparrow\downarrow)\uparrow(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)\uparrow(\uparrow\downarrow)(\uparrow\downarrow)$$

- Neutron scattering only creates the spin-1 excitation = two spinons. It measures the two-spinon gap (spin-1 gap).
  Thermal activation sees single spinon gap.

Neutron scattering spectrum

Neutron dump energy-momentum into the sample creating a few excitations.



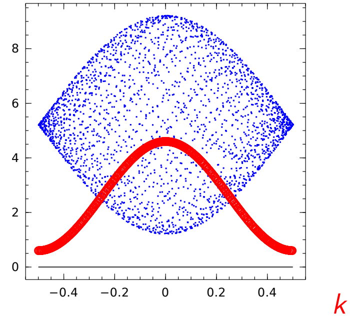
- Without fractionalization, nor trans. symm. breaking

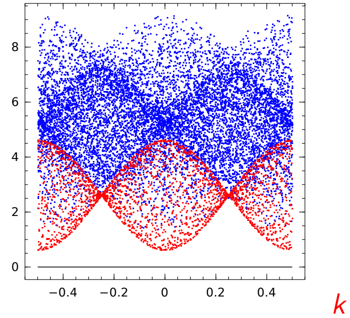
$\epsilon_{\text{spin-1}}(k) = 2.6 + 2\cos(k)$

- With fractionalization and trans. sym. breaking

$\epsilon_{\text{spin-1/2}}(k) = \frac{1}{2}\epsilon(2k)_{\text{spin-1}}$

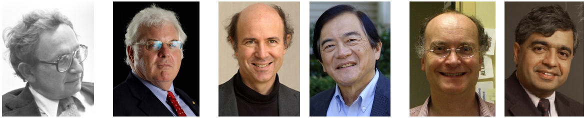
one spin-1 + two spin-1 | two spin-1/2 + four spin-1/2





2D Spin liquid without symmetry breaking (topo. order)

The spin-1 fractionalization into spin-1/2 spinon can happen in 2D spin liquid without translation and $SO(3)$ spin-rotation symmetry breaking:



- On square lattice:

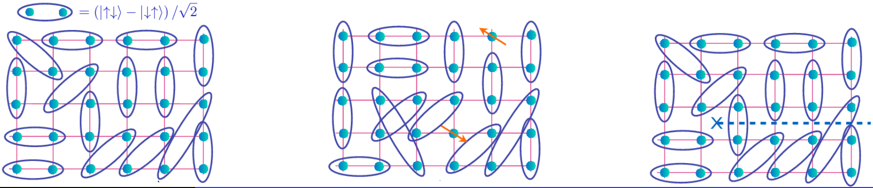
chiral spin liquid $\sum \Psi(RVB)|RVB\rangle \to$ topological order

Kalmeyer-Laughlin PRL **59** 2095 (87); Wen-Wilczek-Zee PRB **39** 11413 (89)

Z_2 **spin liquid** $\sum |RVB\rangle$ (emergent low energy Z_2 gauge theory)

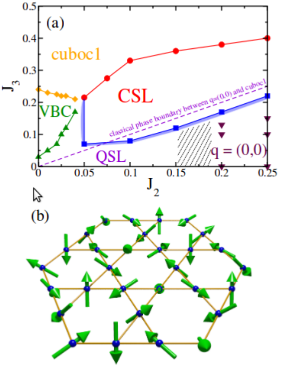
Read-Sachdev PRL **66** 1773 (91); Wen PRB **44** 2664 (91)

Z_2-charge (spin-1/2) = Spinon. Z_2-vortex (spin-0) = Vison. Bound state = fermion (spin-1/2).

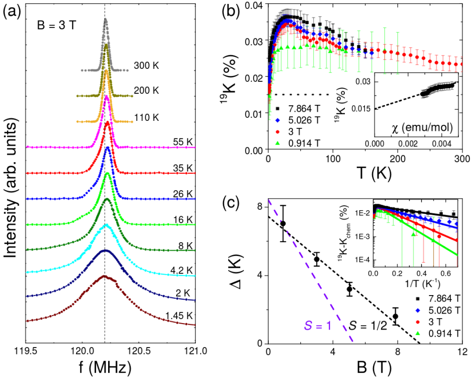


2D Spin liquid without symmetry breaking (topo. order)

- On Kagome lattice:

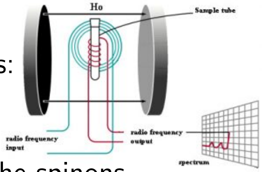


Feng etal arXiv:1702.01658 $Cu_3Zn(OH)_6FBr$



J_1-J_2-J_3 model Gong-Zhu-Balents-Sheng arXiv:1412.1571

- Uniform spin susceptibilty comes from spin excitations: $\chi \sim e^{-\Delta_{\text{spinon}}/k_B T}$. In a strong magnetic field, the activation gap Δ_{spinon} is reduced to $\Delta_{\text{spinon}} - Bgs$. Knowing the g-factor, we can measure the spin s of the spinons.



Duality between 1D boson/spin and 1D fermion systems

To obtain the correct critical theory for the transverse Ising model, we need to use the duality between 1D boson/spin systems and 1D fermion systems.
Duality: Two different theories that describe the same thing.
Two different looking theories that are equivalent.

- If we view down-spin as vacuum and up-spin as a boson, we can view a hard-core boson system as a spin-1/2 system. Now we view a system of hard-core bosons hopping on a line/ring of L sites as a spin-1/2 system. How to write down the spin Hamiltonian to describe such a boson-hopping system?

  $\sigma^{\pm} = (\sigma^x \pm \mathrm{i}\sigma^y)/2$: σ_i^- annihilates (σ_i^+ creates) a boson at site-i, $|\downarrow\rangle = |0\rangle, |\uparrow\rangle = |1\rangle$. $H_{\text{boson-hc}} = \sum_i (-t\sigma_i^+\sigma_{i+1}^- + h.c.)$ describes a hard-core bosons hopping model.

- Similarly, we can also view a system of spin-less fermions on a line/ring of L sites as a spin-1/2 system. How to write down the spin Hamiltonian for such a fermion-hopping system?

Jordan-Wigner transformation on a 1D line of L sites

- $c_i = \sigma_i^+ \prod_{j<i} \sigma_j^z$, $\sigma^\pm = (\sigma^x \pm \mathrm{i}\sigma^y)/2$. One can check that
$$\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0, \quad \{c_i, c_j^\dagger\} = \delta_{ij}, \quad \{A, B\} \equiv AB - BA.$$
$c_i^\dagger, c_i$ create/annihilate a **fermion** at site-i, $|\downarrow\rangle = |0\rangle, |\uparrow\rangle = |1\rangle$
- Mapping between spin/boson chain and fermion chain:
$c_i^\dagger c_i = \sigma_i^- \sigma_i^+ = (-\sigma_i^z + 1)/2 = n_i$, fermion number operator
$c_i^\dagger c_{i+1} = \sigma_i^- \sigma_{i+1}^+ \sigma_i^z = \sigma_i^- \sigma_{i+1}^+$, $\quad c_i c_{i+1} = \sigma_i^+ \sigma_{i+1}^+ \sigma_i^z = -\sigma_i^+ \sigma_{i+1}^+$
- **XY model = fermion model** on an open chain
$H_{\text{fermion}} = \sum_i (-t c_i^\dagger c_{i+1} + h.c.) - \mu n_i \quad \leftrightarrow$
$H_{\text{XY}} = \sum_i (-t\sigma_i^+ \sigma_{i+1}^- + h.c.) + \mu \frac{\sigma_i^z}{2} = \sum_i -\frac{t}{2}(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + \mu \frac{\sigma_i^z}{2}$
- A phase transition in XY model: as we tune μ through $\mu_c = \pm 2t$, the ground state energy density ϵ_μ has a singularity
$\rightarrow$ a phase transition.
How to solve the model exactly to obtain the above result?
The model H_{fermion} or H_{XY} looks not solvable since H's are not a sum of commuting terms.

Make the Hamiltonian into a sum of commuting terms

- The anti-commutation relation

$$\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0, \qquad \{c_i, c_j^\dagger\} = \delta_{ij}$$

is invariant under the unitary transformation of the fermion operators:

$$\tilde{c}_i = U_{ij} c_j : \qquad \{\tilde{c}_i, \tilde{c}_j\} = \{\tilde{c}_i^\dagger, \tilde{c}_j^\dagger\} = 0, \qquad \{\tilde{c}_i, \tilde{c}_j^\dagger\} = \delta_{ij}$$

- Assume the fermions live on a ring. see the homework
Let $\psi_k = \frac{1}{\sqrt{L}} \sum_i \mathrm{e}^{\mathrm{i}\, ki} c_i$ ($k = \frac{2\pi}{L} \times \text{integer}$)

$$H_{\text{fermion}} = \sum_i (-t c_i^\dagger c_{i+1} + h.c.) + g c_i^\dagger c_i = \sum_k \epsilon(k) \psi_k^\dagger \psi_k$$

$$\epsilon(k) = -2t \cos k - \mu, \quad [\psi_k^\dagger \psi_k, \psi_{k'}^\dagger \psi_{k'}] = 0, \quad n_k \equiv \psi_k^\dagger \psi_k = \pm 1.$$

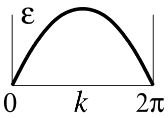
- From the one-body dispersion, we obtain many-body energy spectrum
$E = \sum_k \epsilon(k) n_k,\ K = \sum_k k n_k \bmod \frac{2\pi}{a},\ n_k = 0, 1.$

Majorana fermions and critical point of Ising model

- $\lambda_i^x = \sigma_i^x \prod_{j<i} \sigma_j^z, \quad \lambda_i^y = \sigma_i^y \prod_{j<i} \sigma_j^z$. One can check that
$$(\lambda_i^x)^\dagger = \lambda_i^x,\ (\lambda_i^y)^\dagger = \lambda_i^y;\ \ \{\lambda_i^x, \lambda_j^x\} = \{\lambda_i^y, \lambda_j^y\} = 2\delta_{ij},\ \{\lambda_i^x, \lambda_j^y\} = 0.$$

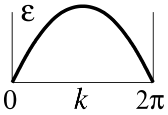
- **Ising model = Majorana-fermion** on a open chain of L sites:
$$\lambda_i^x \lambda_i^y = \mathrm{i}\sigma_i^z, \qquad \lambda_i^y \lambda_{i+1}^x = \sigma_i^y \sigma_{i+1}^x \sigma_i^z = \mathrm{i}\sigma_i^x \sigma_{i+1}^x$$
$$H_{\text{Ising}} = \sum_i -\sigma_i^x \sigma_{i+1}^x - h\sigma_i^z \quad \leftrightarrow \quad H_{\text{fermion}} = \sum_i \mathrm{i}\lambda_i^y \lambda_{i+1}^x + \mathrm{i}\, h \lambda_i^x \lambda_i^y$$
Critical point (gapless point) is at $h = 1$ (not $h = 2$ from meanfield theory): $H_{\text{fermion}}^{\text{critical}} = \sum_I \mathrm{i}\eta_I \eta_{I+1}, \quad \eta_{2i+1} = \lambda_i^x,\ \ \eta_{2i} = \lambda_i^y.$

- In k-space, $\psi_k = \frac{1}{\sqrt{2}} \sum_I \frac{\mathrm{e}^{\mathrm{i}\frac{k}{2}I}}{\sqrt{2L}} \eta_I$, $\frac{k}{2} = \frac{2\pi}{2L} n \in [-\pi, \pi]$:



$$\psi_k^\dagger = \psi_{-k}, \quad \{\psi_k^\dagger, \psi_{k'}\} = \delta_{k-k'} \quad \text{(assume on a ring)}$$
$$H_{\text{fermion}}^{\text{critical}} = \sum_{k\in[-2\pi,2\pi]} 2\mathrm{i}\, \mathrm{e}^{\mathrm{i}\frac{1}{2}k} \psi_{-k}\psi_k = \sum_{k\in[0,2\pi]} \epsilon(k)\psi_k^\dagger \psi_k, \quad \epsilon(k) = 4|\sin\frac{k}{2}|.$$

1D Ising critical point: 1/2 mode of right (left) movers



- The Majorana fermion contain
  a right-moving mode $\epsilon = vk$ and
  a left-moving modes. $\epsilon = -vk$
- Thermal energy density (for a right moving mode):
  $$\epsilon_T = \int_0^{+\infty} \frac{\mathrm{d}k}{2\pi} \frac{vk}{\mathrm{e}^{vk/k_B T} + 1} = \frac{k_B^2 T^2}{2\pi v} \int_0^{+\infty} \mathrm{d}x \frac{x}{\mathrm{e}^x + 1} = \frac{k_B^2 T^2}{v} \frac{\pi}{24}$$
  where $\int_0^{+\infty} \mathrm{d}x \frac{x}{\mathrm{e}^x+1} = \frac{\pi^2}{12}$
- Specific heat
  $$c_T = \frac{\partial \epsilon_T}{\partial T} = \frac{1}{2} k_B \frac{k_B T}{v} \frac{\pi}{6}$$
  Central charge $c = 1/2$ for right (left) movers.
- On a ring of size L and at critical point: the ground state energy has a form $E = \epsilon L + \frac{2\pi v}{L}(-\frac{c}{24})$, where c in the "Casimir term" (the $1/L$ term) is also the central charge.
  Do we have a similar result for an open Line?

A story about central charge c (conformal field theory)

- Central charge is a property of 1D gapless system with a finite and unique velocity. $c = c_L + c_R = 0$ for gapped systems.
- It has an additive property: $A \boxtimes_{\text{stacking}} B = C \to c_A + c_B = c_C$
- It measures how many low energy excitation are there.
  Specific heat (heat capacity per unit length) $C = c\frac{\pi}{6}\frac{T}{v}$

A story about central charge c (conformal field theory)

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\- Why ground state energy $E = \rho_\epsilon L - \frac{c}{24}\frac{2\pi}{L}$ sees central charge ($v = 1$)?
Partition function: $Z(\beta, L) = \mathrm{Tr}(\mathrm{e}^{-\beta H}) = \mathrm{e}^{-\beta L \rho_\epsilon - \frac{2\pi\beta}{L}\frac{c}{24}}|_{\beta\to\infty}$

- A magic: emergence of $O(2)$ symmetry in space-(imaginary-)time

$$Z(\beta, L) = Z(L, \beta), \qquad \text{have used } v = 1.$$

This allows us to find $Z(\beta, L) = \mathrm{e}^{-\beta L \rho_\epsilon - \frac{2\pi L}{\beta}\frac{c}{24}}|_{L\to\infty}$

Free energy density $f = \rho_\epsilon - \frac{2\pi}{(\beta)^2}\frac{c}{24}$

$$= \rho_\epsilon - 2\pi T^2 \frac{c}{24}$$

Specific heat $C = -T\frac{\partial^2 F}{\partial T^2} = T\frac{\pi}{6}c$

Belavin-Polyakov-Zamolodchikov NPB 241,333(84); Ginsparg hep-th/9108028

The neutron scattering and spectral function (Ising model)

Assume the neutron spin couples to Ising spin via $S_i^z \sim \sigma_i^z$ (no S^z-spin flip, but scattering flips $S^{x,y}$). After scattering, the neutron dump something to the system $|\Psi\rangle \to \sigma_i^z|\Psi\rangle$. What is the scattering spectrum? **The spectra function of σ_i^z**:

$$I(E,K) = \langle\Psi|\sigma_i^z \delta(\hat{H}-E)\delta(\hat{K}-K)\sigma_i^z|\Psi\rangle$$

$$\sigma_i^z = \mathrm{i}\eta_{2i}\eta_{2i+1} = \frac{2\mathrm{i}}{L}\sum_{k_1,k_2} \mathrm{e}^{\mathrm{i}\,k_1 i}\mathrm{e}^{\mathrm{i}\,k_2(i+\frac{1}{2})}\psi_{k_1}\psi_{k_2}$$

$$I(E,K) = \frac{4}{L^2}\langle\Psi|\sum_{k_1,k_2} \mathrm{e}^{\mathrm{i}\,k_1 i}\mathrm{e}^{\mathrm{i}\,k_2(i+\frac{1}{2})}\psi_{k_1}\psi_{k_2}\delta(\epsilon_{k_1'}+\epsilon_{k_2'}-E)$$

$$\delta(k_1'+k_2'-K)\sum_{k_1',k_2'} \mathrm{e}^{-\mathrm{i}\,k_1' i}\mathrm{e}^{-\mathrm{i}\,k_2'(i+\frac{1}{2})}\psi_{k_2'}^\dagger\psi_{k_1'}^\dagger|\Psi\rangle$$

$$= \frac{4}{L^2}\sum_{k_1,k_2\in[0,2\pi]} \delta(\epsilon_{k_1}+\epsilon_{k_2}-E)\delta(k_1+k_2-K)(1-\mathrm{e}^{\mathrm{i}\,\frac{1}{2}(k_1-k_2)})$$

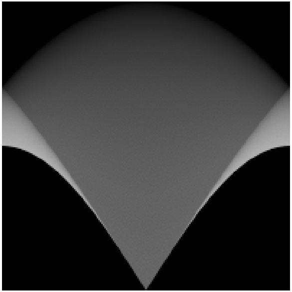
The neutron scattering and spectral function (Ising model)

$$I(E,K)=4\int_0^{2\pi}\frac{\mathrm{d}k_1\,\mathrm{d}k_2}{(2\pi)^2}\delta(\epsilon_{k_1}+\epsilon_{k_2}-E)\delta(k_1+k_2-K)(1-\cos\frac{k_1-k_2}{2})$$

$$I_0(E,K)=4\int_0^{2\pi}\frac{\mathrm{d}k_1\,\mathrm{d}k_2}{(2\pi)^2}\delta(\epsilon_{k_1}+\epsilon_{k_2}-E)\delta(k_1+k_2-K)$$

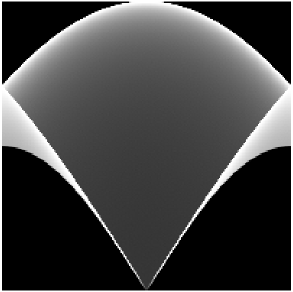
where $\epsilon_k=4|\sin\frac{k}{2}|$.

$I(E,K)$



$-\pi$ K π

$I_0(E,K)$: two-fermion density of states



$-\pi$ K π

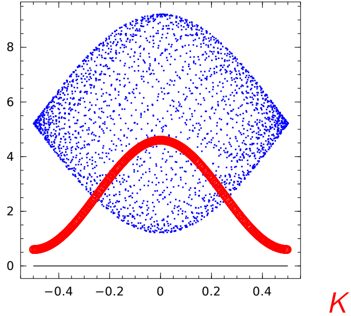
- What is the spectral function for σ_i^x? for $\sigma_i^x\sigma_j^x$? Why σ_i^x is hard?

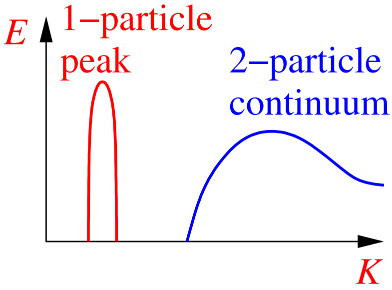
A general picture of specture function

We can understand the spectral function of an operator O_x by writing it in terms of quasiparticle creating/annihilation operators

$$
\begin{aligned}
O_i &= C_1 a_i^\dagger + C_2 a_i^\dagger a_{i+1}^\dagger + \cdots \\
&= C_1 \int \mathrm{d}k\ a_k^\dagger + + C_2 \int \mathrm{d}k_1\, \mathrm{d}k_2\ a_{k_1}^\dagger a_{k_2}^\dagger\, \mathrm{e}^{-\mathrm{i}[k_1 i + k_2(i+1)]} + \ldots
\end{aligned}
$$

Assume one-particle spectrum to be $\epsilon(k) = 2.6 + 2\cos(k) \to$
Two-particle spectrum will be $E = \epsilon(k_1) + \epsilon(k_2),\ K = k_1 + k_2$





Specture function and time-ordered correlation functions

- Consider a 0d system with ground state $|0\rangle$ with energy $E_0 = 0$. An operator O creates excitations, and have a spectral function
$$I(\omega) = \langle 0|O^\dagger \delta(\hat H - \omega) O|0\rangle.$$
- Time-ordered correlation function of $O(t) = \mathrm{e}^{\mathrm{i}\hat H t} O \mathrm{e}^{-\mathrm{i}\hat H t}$:

$$G(t) = \mathrm{i}\langle 0|\mathcal{T}[O(t)O(0)]|0\rangle = \mathrm{i}\begin{cases}\langle 0|O(t)O(0)|0\rangle, & t>0\\ \langle 0|O(0)O(t)|0\rangle, & t<0\end{cases}$$

$$= \mathrm{i}\begin{cases}\langle 0|O\mathrm{e}^{-\mathrm{i}\hat H t}O|0\rangle, & t>0\\ \langle 0|O\mathrm{e}^{\mathrm{i}\hat H t}O|0\rangle, & t<0\end{cases} = \mathrm{i}\begin{cases}\int_0^{+\infty} \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega t} I(\omega), & t>0\\ \int_0^{+\infty} \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega t} I(\omega), & t<0\end{cases}$$

$$G(\omega) = \int \mathrm{d}t\, G(t)\mathrm{e}^{\mathrm{i}\omega t} = \mathrm{i}\int_0^{+\infty}\mathrm{d}t\int_0^{+\infty}\mathrm{d}\omega' \Big(\mathrm{e}^{-\mathrm{i}(\omega'-\omega-\mathrm{i}0^+)t} I(\omega') - \mathrm{e}^{-\mathrm{i}(\omega'+\omega-\mathrm{i}0^+)t} I(\omega')\Big)$$

$$= \int_0^{+\infty}\mathrm{d}\omega'\Big(\frac{I(\omega')}{\omega'-\omega-\mathrm{i}0^+} - \frac{I(\omega')}{\omega'+\omega-\mathrm{i}0^+}\Big) = \int_{-\infty}^{+\infty}\mathrm{d}\omega' \frac{I(|\omega'|)}{\omega'-\omega-\mathrm{i}0^+\,\mathrm{sgn}\omega'}$$

$$I(\omega) = \frac{\mathrm{sgn}(\omega)}{\pi}\mathrm{Im}\, G(\omega).$$ Adding $\mathrm{i}0^+$ to regulate the integral $\int_0^{+\infty}\mathrm{d}t$

- In higher dimensions: $G(t,x) \to G(\omega,k) \to I(\omega,k) = \frac{\mathrm{sgn}(\omega)}{\pi}\mathrm{Im}\, G(\omega,k)$

The neutron scattering and spectral function (XY model)

1D XY model (superfuild of bosons) = 1D non-interacting fermions

$H_{\text{XY}} = \sum_i -\frac{t}{2}(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) - \mu \frac{\sigma_i^z}{2} \leftrightarrow H_{\text{f}} = \sum_i (t c_i^\dagger c_{i+1} + h.c.) - \mu n_i$

Let us assume the neutron coupling is $S_i^z \sim \sigma_i^z$ (*ie* neutrons see the boson density) $\to$ Spectral function of operator $\sigma_i^z = c_i^\dagger c_i$ (adding a particle-hole pair)

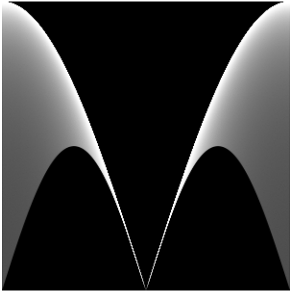
$$
\begin{aligned}
I(E,K) &= \langle \Psi | c_i^\dagger c_i \delta(\hat{H} - E) \delta(\hat{K} - K) c_i^\dagger c_i | \Psi \rangle \\
&= \frac{1}{L^2} \langle \Psi | \sum_{k_1,k_2} \mathrm{e}^{\mathrm{i}\, k_1 i} \mathrm{e}^{\mathrm{i}\, k_2 i} \psi_{k_1}^\dagger \psi_{k_2} \delta(-\epsilon_{k_1'} + \epsilon_{k_2'} - E) \\
&\qquad \delta(-k_1' + k_2' - K) \sum_{k_1',k_2'} \mathrm{e}^{-\mathrm{i}\, k_1' i} \mathrm{e}^{-\mathrm{i}\, k_2' i} \psi_{k_2'}^\dagger \psi_{k_1'} | \Psi \rangle \\
&= \int_{\epsilon_{k_1} < 0,\ \epsilon_{k_2} > 0} \frac{\mathrm{d}k_1\, \mathrm{d}k_2}{(2\pi)^2} \delta(-\epsilon_{k_1} + \epsilon_{k_2} - E) \delta(-k_1 + k_2 - K)
\end{aligned}
$$

where $\epsilon_k = 2t \cos k - \mu$ and $c_i = \frac{1}{\sqrt{L}} \sum_k \mathrm{e}^{\mathrm{i}\, ki} \psi_k$

The neutron scattering and spectral function (XY model)

Spectral function of $n_i \sim \sigma_i^z$ for the superfluid/XY-model

For $\mu = 0, \ \langle \sigma_i^z \rangle = 0$



$-\pi$ K π

For $\mu = -1, \ \langle \sigma_i^z \rangle \neq 0$

$-\pi$ K π

Particle-hole spactral function. In additional to the low energy excitations near $k = 0$, why are there low energy excitations at large $K_\pm = \pm 2\pi n$? $K_\pm$ only depend on boson density n! What is the single particle spectral function of σ_i^+? $\sigma_i^+ = c_i^\dagger \prod_{j<i}(2c_j^\dagger c_j - 1)$

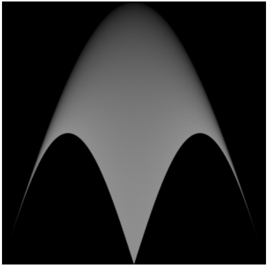
The neutron scattering and spectral function (XY model)

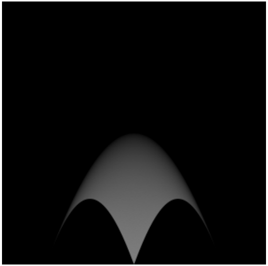
Particle-particle spectral function of $\sigma_i^+\sigma_{i+1}^+$ (adding two bosons)

$$
\begin{aligned}
I(E,K) &= \langle\Psi|c_{i+1}c_i\delta(\hat H - E)\delta(\hat K - K)c_i^\dagger c_{i+1}^\dagger|\Psi\rangle \\
&= \frac{1}{L^2}\langle\Psi|\sum_{k_1,k_2}\mathrm{e}^{\mathrm{i}\,k_1(i+1)}\mathrm{e}^{\mathrm{i}\,k_2 i}\psi_{k_1}\psi_{k_2}\delta(\epsilon_{k_1'}+\epsilon_{k_2'}-E) \\
&\qquad \delta(k_1'+k_2'-K)\sum_{k_1',k_2'}\mathrm{e}^{-\mathrm{i}\,k_1'(i+1)}\mathrm{e}^{-\mathrm{i}\,k_2' i}\psi_{k_2'}^\dagger\psi_{k_1'}^\dagger|\Psi\rangle
\end{aligned}
$$

$$
= \int_{\substack{\epsilon_{k_1}>0\\ \epsilon_{k_2}>0}}\frac{\mathrm{d}k_1\,\mathrm{d}k_2}{(2\pi)^2}\delta(\epsilon_{k_1}+\epsilon_{k_2}-E)\delta(k_1+k_2-K)[1-\cos(k_1-k_2)]
$$

$\mu = 0$ and $\mu = -1$ 2-particle spectral function





XY model for superfluid: dynamical variational approach

Compute single-particle spectral function using an approximation

We are going to use the approximated variational approach for XY model (not bad for superfluid phase. See also prob. 4.2):
$H = -\sum_i J(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y) + h\sigma_i^z)$.
Trial wave function $|\Psi_{\phi_i}\rangle = \otimes_i|\phi_i\rangle$,
where $|\phi_i\rangle = \frac{|\uparrow\rangle+\phi_i|\downarrow\rangle}{\sqrt{1+|\phi_i|^2}}$, $\langle\sigma_i^+\rangle = \frac{\phi_i}{1+|\phi_i|^2}$.

- Average energy $\bar{H} = -\sum_i \left[2J\frac{\phi_i\phi_{i+1}^*+h.c.}{(1+|\phi_i|^2)(1+|\phi_{i+1}|^2)} + h\frac{1-|\phi_i|^2}{1+|\phi_i|^2}\right]$

  Geometric phase term $\langle\phi_i|\frac{\mathrm{d}}{\mathrm{d}t}|\phi_i\rangle = \frac{\phi_i^*\dot{\phi}_i}{1+|\phi_i|^2} + \frac{\mathrm{d}}{\mathrm{d}t}\#$

  Phase space Lagrangian in symmetry breaking phase (up to φ_i^2)
  ($\phi_i = \bar{\phi} + \varphi_i$ for $J > 0$ or $\phi_i = \bar{\phi}(-)^i + \varphi_i$ for $J < 0$)

$$L = \langle\Phi_{\phi_i}|\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - H|\Phi_{\phi_i}\rangle = \sum_i \mathrm{i}\phi_i^*\dot{\phi}_i + 2J(\phi_i\phi_{i+1}^* + h.c.) - 2h|\phi_i|^2 - g|\phi_i|^4$$

$$= \sum_i \mathrm{i}\varphi_i^*\dot{\varphi}_i + 2J(\varphi_i\varphi_{i+1}^* + h.c.) - 2h\varphi_i\varphi_i^* - g\bar{\phi}^2[4\varphi_i\varphi_i^* + \underbrace{\varphi_i^2 + (\varphi_i^*)^2}_{\text{symm breaking}}]$$

with $g\bar{\phi}^2 = 2|J| - h$.

Quantum XY model

Quantization:

$$[\varphi_i, \varphi_j^\dagger] = \delta_{ij}, \quad \varphi_i = \frac{1}{\sqrt{L}} \sum_k \mathrm{e}^{\mathrm{i}\, ki} a_k, \quad [a_k, a_q^\dagger] = \delta_{kq}$$

$$H = \sum_i -2J(\varphi_i \varphi_{i+1}^\dagger + h.c.) + 2h\varphi_i^\dagger \varphi_i + (2|J| - h)(4\varphi_i^\dagger \varphi_i + \varphi_i\varphi_i + \varphi_i^\dagger \varphi_i^\dagger)$$

$$= \sum_k (-4J\cos k + 8|J| - 2h) a_k^\dagger a_k + (2|J| - h)(a_k a_{-k} + a_k^\dagger a_{-k}^\dagger)$$

$$= \sum_{k\in[0,\pi]} \begin{pmatrix} a_k^\dagger & a_{-k} \end{pmatrix} \begin{pmatrix} -4J\cos k + 8|J| - 2h & 2(2|J|-h) \\ 2(2|J|-h) & -4J\cos k + 8|J| - 2h \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix}$$

$$= \sum_{k\in[0,\pi]} \begin{pmatrix} a_k^\dagger & a_{-k} \end{pmatrix} \begin{pmatrix} \epsilon_k & \Delta \\ \Delta & \epsilon_k \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix}, \quad \begin{aligned} \epsilon_k &= -4J\cos k + 8|J| - 2h, \\ \Delta &= 2(2|J| - h). \end{aligned}$$

To diagonalize the above Hamiltonian, let

$$\begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} = U \begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix}, \quad U = \begin{pmatrix} u_k & -v_k \\ -v_k & u_k \end{pmatrix}, \quad U \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $u_k^2 - v_k^2 = 1$

Quantum XY model

$$H = \sum_{k\in[0,\pi]} \begin{pmatrix} a_k^\dagger & a_{-k} \end{pmatrix} \begin{pmatrix} \epsilon_k & \Delta \\ \Delta & \epsilon_k \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix}$$

$$U \begin{pmatrix} \epsilon_k & \Delta \\ \Delta & \epsilon_k \end{pmatrix} U = \begin{pmatrix} (u^2+v^2)\epsilon_k - 2uv\Delta & (u^2+v^2)\Delta - 2uv\epsilon_k \\ (u^2+v^2)\Delta - 2uv\epsilon_k & (u^2+v^2)\epsilon_k - 2uv\Delta \end{pmatrix}$$

$$= \begin{pmatrix} E_k & 0 \\ 0 & E_k \end{pmatrix}, \qquad E_k = \sqrt{\epsilon_k^2 - \Delta^2}$$

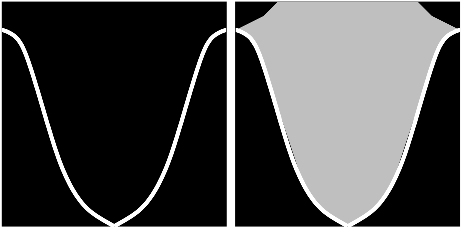
$$u^2 + v^2 = \frac{\epsilon_k}{E_k}, \qquad 2uv = \frac{\Delta}{E_k},$$

$$u = \sqrt{\frac{\frac{\epsilon_k}{E_k}+1}{2}}, \qquad v = \sqrt{\frac{\frac{\epsilon_k}{E_k}-1}{2}}$$

$$H = \sum_k b_k^\dagger \underbrace{\sqrt{(-4J\cos k + 8|J| - 2h)^2 - (4|J| - 2h)^2}}_{\sqrt{\epsilon_k^2-\Delta^2}=E_k\to 0|_{k\to 0},\quad \text{spin-wave dispersion}} b_k$$

The spectral function – XY model (only for $\langle\sigma^+\rangle = \bar\phi$)

- Spectral function for $\sigma^+ \sim \bar\phi + \varphi_i^\dagger$, and $(\sigma^+)^2 \sim \bar\phi^2 + 2\bar\phi\varphi_i^\dagger + (\varphi_i^\dagger)^2$



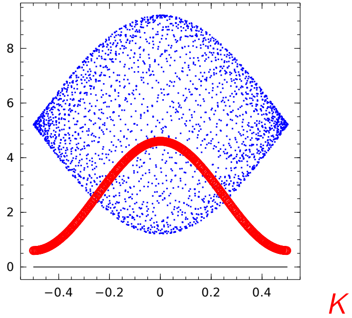
$$\varphi_i^\dagger = \frac{1}{\sqrt{L}}\sum_k \mathrm{e}^{-\mathrm{i}\,ki} a_k^\dagger$$
$$= \frac{1}{\sqrt{L}}\sum_k \mathrm{e}^{-\mathrm{i}\,ki}(u_k b_k^\dagger - v_k b_{-k})$$
$$I(E,K) \sim u_K^2 \delta(E_K - E) = \frac{\frac{\epsilon_k}{E_k}+1}{2}\delta(E_K - E) \to \infty|_{k\to 0}$$

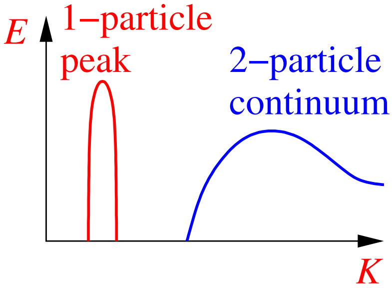
- Spectral function for $n_i = \frac{\sigma_i^z - 1}{2} \sim \sigma_i^x \sim \varphi_i + \varphi_i^\dagger$

$$\varphi_i + \varphi_i^\dagger = \frac{1}{\sqrt{L}}\sum_k \mathrm{e}^{-\mathrm{i}\,ki}(a_{-k} + a_k^\dagger)$$
$$= \frac{1}{\sqrt{L}}\sum_k \mathrm{e}^{-\mathrm{i}\,ki}(u_k b_{-k} - v_k b_k^\dagger + u_k b_k^\dagger - v_k b_{-k})$$
$$I(E,K) \sim (u_K - v_K)^2 \delta(E_K - E) = \frac{E_k}{\epsilon_k + \Delta}\delta(E_K - E) \to 0|_{k\to 0}$$

The spectral function – XY model (only for $\langle\sigma^+\rangle = \bar\phi$)

The following picture work in higher dimension since $\langle\sigma_i^+\rangle = \bar\phi$ (symmetry breaking) $\langle\sigma_i^+\sigma_j^-\rangle \sim$ *const*. for large $|i-j|$





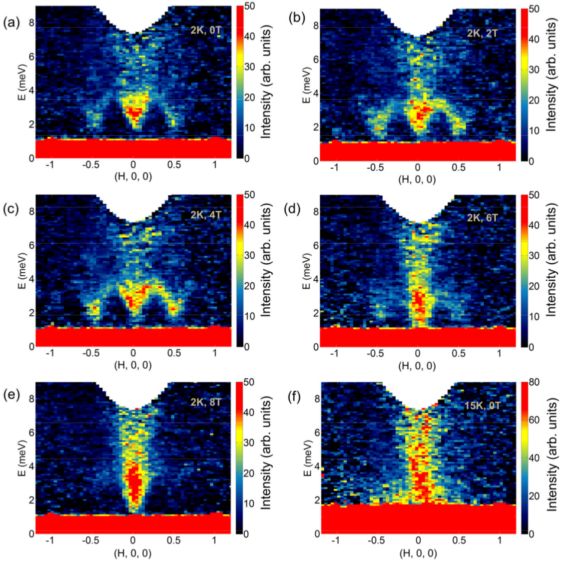
But does not quite work in 1 dimension (or 1+1 dimensions) since $\langle\sigma_i^+\rangle = 0$ (no symmetry breaking).
We only have a nearly symmetry breaking

$$\langle\sigma_i^+\sigma_j^-\rangle \sim \frac{1}{|i-j|^\alpha} \text{ for large } |i-j|$$

Neutron scattering spectrum for 2-dimensional α-$RuCl_3$

Banerjee etal arXiv:1706.07003

- Spin-1/2 on honeycomb lattice with strong spin-orbital coupling.
- Spin ordered phase below 8T, spin liquid above 8T
- Magnetic field:
  (a-e) $B: 0, 2, 4, 6, 8T$
  (a-e) $T = 2K$
  (f) $T = 2K, B = 0T$



1d field theory to study no $U(1)$ symmetry breaking in 1D

Phase space Lagrangian in "symmetry breaking phase" of 1D XY model: $\phi_i = (\bar\phi + q_i)\mathrm{e}^{-\mathrm{i}\theta_i}$, $\bar\phi^2 = \frac{2J-h}{g}$, near the transition $\bar\phi \sim 0$

$$L = \sum_i \mathrm{i}\phi_i^* \dot\phi_i + 2J(\phi_i\phi_{i+1}^* + h.c.) - 2h|\phi_i|^2 - g|\phi_i|^4$$
$$\approx \sum_i (\bar\phi + q_i)^2\dot\theta_i + \frac{1}{2}\partial_t(\bar\phi + q_i)^2$$
$$+ 2J|\bar\phi|^2(\mathrm{e}^{\mathrm{i}(\theta_i - \theta_{i+1})} + h.c.) - 4(2J - h)q_i^2,$$

where we kept up to q_i^2 terms. The total derivative term $\frac{1}{2}\partial_t(\bar\phi + q_i)^2$ can be dropped. The total "derivative" term $\bar\phi^2\dot\theta_i$ cannot be dropped since it is not a total derivative $\bar\phi^2\dot\theta_i = \mathrm{i}\bar\phi^2\mathrm{e}^{\mathrm{i}\theta}\partial_t\mathrm{e}^{-\mathrm{i}\theta}$.

1d field theory to study no $U(1)$ symmetry breaking in 1D

After dropping $q_i^2\dot\theta_i$ term, we obtain

$$L = \sum_i (\bar\phi^2 + 2\bar\phi q_i)\dot\theta_i - 2J|\bar\phi|^2(\theta_i - \theta_{i+1})^2 - 4(2J-h)q_i^2$$

$$= \int \mathrm{d}x \ [\bar\phi^2 + \underbrace{\frac{2\bar\phi}{a}q(x)}_{\partial_x\varphi/2\pi}]\dot\theta(x) - 2J|\bar\phi|^2 a[\partial_x\theta(x)]^2 - \frac{4(2J-h)}{a}q^2(x)$$

$$= \int \mathrm{d}x \ \frac{1}{2\pi}\partial_x\varphi\partial_t\theta - \frac{1}{4\pi}V_1(\partial_x\theta)^2 - \frac{1}{4\pi}V_2(\partial_x\varphi)^2 + \frac{\bar\phi^2}{a}\partial_t\theta$$

where $V_1 = \frac{8\pi J(2J-h)a}{g}, \quad V_2 = \frac{ga}{\pi}$.

- Momentum of uniform $\theta(x)$: $\int \mathrm{d}x \frac{\partial_x\varphi}{2\pi} = \frac{\Delta\varphi}{2\pi} = int. \to \varphi$ also live on S^1: $\varphi \sim \varphi + 2\pi$

Both θ and φ are compact angular fields living on S^1.

1d field theory with two angular fields

- Let $\varphi_1 = \theta$ and $\varphi_2 = \varphi$, we can rewrite that above as phase space Lagrangian as

$$L = \int \mathrm{d}x \; \frac{2}{4\pi}\partial_x\varphi_2\partial_t\varphi_1 - \frac{1}{4\pi}V_1(\partial_x\varphi_1)^2 - \frac{1}{4\pi}V_2(\partial_x\varphi_2)^2 + \frac{\bar\varphi^2}{a}\partial_t\varphi_1,$$

which has the following general form

$$L = \int \mathrm{d}x \; \frac{K_{IJ}}{4\pi}\partial_x\varphi_I\partial_t\varphi_J - \frac{V_{IJ}}{4\pi}\partial_x\varphi_I\partial_x\varphi_J, \;\; \varphi_I \sim \varphi_I + 2\pi, K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- **A very generic 1+1D bosonic model**: Compact fields $\phi_I \sim \phi_I + 2\pi$. V is symmetric and positive definite. K is a symmetric integer matrix.

- Positive eigenvalues of K $\to$ left movers.
  Negative eigenvalues of K $\to$ right movers. (See next page)
- The model is **chiral** if the right and left movers are not symmetric.
- For bosonic system, the diagonal of K are all even. For fermionic system, some diagonal of K are odd even.
- The field theory is not realizable by lattice model if $K \not\cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *ie* has gravitational anomalies.

1d field theory: right movers and left movers

- Introduce $\begin{pmatrix}\theta\\ \varphi\end{pmatrix} = U\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$, we can diagonaliz K, V simultaneously: $K \to U^\top K U,\ V \to U^\top V U$. Let $U = U_1 U_2$.

- We first use U_1 to transform $V \to U_1^\top V U_1 = \text{id}$. $K \to U_1^\top K U_1$.
- We then use orthorgonal U_2 to transform $U_1^\top K U_1 \to U_2^\top U_1^\top K U_1 U_2 = \text{Diagonal}(\kappa_1, -\kappa_2, \cdots)$ and $U_1^\top V U_1 = \text{id} \to U_2^\top U_1^\top V U_1 U_2 = \text{id}$.

- For our case of $K = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, we find $U = \begin{pmatrix}(2V_1)^{-1/2} & (2V_1)^{-1/2}\\ (2V_2)^{-1/2} & -(2V_2)^{-1/2}\end{pmatrix}$.

$K \to \begin{pmatrix}\kappa & 0\\ 0 & -\kappa\end{pmatrix},\ \kappa = (V_1 V_2)^{-1/2},\ V \to \text{id}$, and

$$L = \int \mathrm{d}x\ \frac{1}{2\pi}\partial_x\varphi\partial_t\theta - \frac{1}{4\pi}V_1(\partial_x\theta)^2 - \frac{1}{4\pi}V_2(\partial_x\varphi)^2 + \underbrace{\frac{\bar{\phi}^2}{a}\partial_t\theta}_{\text{dropped}}$$

$$= \int \mathrm{d}x\ \frac{1}{4\pi}(\kappa\partial_x\phi_1\partial_t\phi_1 - \partial_x\phi_1\partial_x\phi_1) + \frac{1}{4\pi}(-\kappa\partial_x\phi_2\partial_t\phi_2 - \partial_x\phi_2\partial_x\phi_2)$$

- ϕ_1 and ϕ_2 are not really decoupled, since their compactness are mixed.

1d field theory – chiral boson model

$$L = \int \mathrm{d}x \ \frac{\kappa}{4\pi}\partial_x\phi_1(\partial_t\phi_1 - v\partial_x\phi_1) - \frac{\kappa}{4\pi}\partial_x\phi_2(\partial_t\phi_2 + v\partial_x\phi_2)$$

EOM: $\partial_t\phi_1 - v\partial_x\phi_1 = 0$ and $\partial_t\phi_2 + v\partial_x\phi_2 = 0$ $(v = 1/\kappa)$
$\to$ $\phi_1(x + vt)$ is left-mover. $\phi_2(x - vt)$ is right-mover.

- Consider only right-movers ($\phi(x) = \sum_n \mathrm{e}^{-\mathrm{i}\,kx}\phi_n,\ k = k_0 n,\ k_0 = \frac{2\pi}{L}$)

$$L = -\int \mathrm{d}x \ \frac{\kappa}{4\pi}\partial_x\phi(\partial_t\phi + v\partial_x\phi) \qquad \text{(consider only } n \neq 0 \text{ terms)}$$

$$= \sum_{n\neq 0} -\frac{\kappa L}{4\pi}(-\mathrm{i}\,k)\phi_n(\dot\phi_{-n} + \mathrm{i}\,vk\phi_{-n}) = \sum_{n>0} \mathrm{i}\,n\kappa\phi_n(\dot\phi_{-n} + \mathrm{i}\,vk\phi_{-n})$$

Quantize $[x, p] = \mathrm{i}$: $[\phi_{-n}, \mathrm{i}\,n\kappa\phi_n] = \mathrm{i}$, $H = \sum_{n>0} vkn\kappa\phi_n\phi_{-n}$
Let $a_n^\dagger = \sqrt{n\kappa}\phi_n \ \to \ a_n = \sqrt{n\kappa}\phi_{-n}$

$$[a_n, a_n^\dagger] = 1, \quad H = \sum_{n>0} vk\frac{a_n^\dagger a_n + a_n a_n^\dagger}{2} = \sum_{n>0} vk(a_n^\dagger a_n + \frac{1}{2}).$$

Time-ordered correlation function

- Equal time correlation $\langle 0|O(x)O(y)|0\rangle \equiv \langle O(x)O(y)\rangle$
- Time dependent operator $O(t) = \mathrm{e}^{\mathrm{i}Ht} O \mathrm{e}^{-\mathrm{i}Ht}$ so that

$$\langle \Phi'|O(t)|\Phi\rangle = \langle \Phi'(t)|O|\Phi(t)\rangle,$$

where $|\Phi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}|\Phi\rangle$, $|\Phi'(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}|\Phi'\rangle$. We find

$$a_n^\dagger(t) = \mathrm{e}^{\mathrm{i}vkt} a_n^\dagger, \qquad \phi_n(t) = \mathrm{e}^{\mathrm{i}vkt}\phi_n,$$

$$\phi(x,t) = \sum_n \mathrm{e}^{-\mathrm{i}k(x-vt)}\phi_n, \qquad k = \frac{2\pi}{L}n.$$

- Time-ordered correlation

$$-\mathrm{i}\,G(x-y,t) = \langle \mathcal{T}[\phi(x,t)\phi(y,0)]\rangle = \begin{cases} \langle \phi(x,t)\phi(y,0)\rangle, & t>0 \\ \langle \phi(y,0)\phi(x,t)\rangle, & t<0 \end{cases}$$

For anti-commuting operators (to make $G(x,t)$ a continuous function of x, t away from $(x,t) = (0,0)$)

$$-\mathrm{i}\,G(x-y,t) = \langle \mathcal{T}[\psi(x,t)\tilde{\psi}(y,0)]\rangle = \begin{cases} \langle \psi(x,t)\tilde{\psi}(y,0)\rangle, & t>0 \\ -\langle \tilde{\psi}(y,0)\psi(x,t)\rangle, & t<0 \end{cases}$$

Time ordered correlation function of chiral field $\phi(x,t)$

- For $t > 0$ $(k = nk_0,\ k_0 = \frac{2\pi}{L})$

$$\langle\phi(x,t)\phi(0,0)\rangle = \sum_{n_1,n_2} \mathrm{e}^{-\mathrm{i}\,k_1(x-vt)}\langle\phi_{n_1}\phi_{n_2}\rangle = \sum_{n_2>0} \mathrm{e}^{\mathrm{i}\,k_2(x-vt)}\underbrace{\langle\phi_{-n_2}\phi_{n_2}\rangle}_{\frac{a_{n_2}}{\sqrt{n_2\kappa}}\frac{a^\dagger_{n_2}}{\sqrt{n_2\kappa}}}$$

$$= \sum_{n=1}^{\infty} \mathrm{e}^{\mathrm{i}\,2\pi\frac{x-vt}{L}n}\frac{1}{n\kappa} = -\frac{1}{\kappa}\log(1-\mathrm{e}^{\mathrm{i}\,2\pi\frac{x-vt}{L}})$$

since $\sum_{n=1}^{\infty}\mathrm{e}^{\alpha n}\frac{1}{n} = -\log(1-\mathrm{e}^{\alpha}),\ \ \mathrm{Re}(\alpha) < 0$.

- For $t < 0$

$$\langle\phi(0,0)\phi(x,t)\rangle = \sum_{n_1,n_2} \mathrm{e}^{-\mathrm{i}\,k_1(x-vt)}\langle\phi_{n_2}\phi_{n_1}\rangle = \sum_{n_1>0} \mathrm{e}^{-\mathrm{i}\,k_1(x-vt)}\langle\phi_{-n_1}\phi_{n_1}\rangle$$

$$= \sum_{n=1}^{\infty} \mathrm{e}^{-\mathrm{i}\,2\pi\frac{x-vt}{L}n}\frac{1}{n\kappa} = -\frac{1}{\kappa}\log(1-\mathrm{e}^{-\mathrm{i}\,2\pi\frac{x-vt}{L}})$$

Correlation function of vertex operator $\mathrm{e}^{\mathrm{i}\alpha\phi}$

- Normal ordering $(\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{\frac{1}{2}[A,B]} \mathrm{e}^{A+B})$ $\quad [\phi_n, \phi_{-n}] = \frac{1}{\kappa n},\ n > 0$

$$: \mathrm{e}^{\mathrm{i}\alpha\phi(x,t)} := \underbrace{\mathrm{e}^{\mathrm{i}\alpha \sum_{n>0} \mathrm{e}^{\mathrm{i}\,k(x-vt)}\phi_n}}_{\text{creation}} \underbrace{\mathrm{e}^{\mathrm{i}\alpha \sum_{n<0} \mathrm{e}^{\mathrm{i}\,k(x-vt)}\phi_n}}_{\text{annihilation}}$$

$$= \mathrm{e}^{-\frac{\alpha^2}{2}[\sum_{n>0} \mathrm{e}^{\mathrm{i}\,k(x-vt)}\phi_n, \sum_{n<0} \mathrm{e}^{\mathrm{i}\,k(x-vt)}\phi_n]} \mathrm{e}^{\mathrm{i}\phi(x,t)} = \underbrace{\mathrm{e}^{\frac{\alpha^2}{2\kappa}\sum_n \frac{1}{n}}}_{\sim (\frac{L}{a})^{\frac{\alpha^2}{2\kappa}}} \mathrm{e}^{\mathrm{i}\phi(x,t)}$$

- Correlation function $(\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{[A,B]} \mathrm{e}^B \mathrm{e}^A)$

$$\langle : \mathrm{e}^{\mathrm{i}\alpha\phi(x,t)} :: \mathrm{e}^{-\mathrm{i}\alpha\phi(0,0)} : \rangle = \langle \mathrm{e}^{\mathrm{i}\alpha\phi_>(x,t)} \mathrm{e}^{\mathrm{i}\alpha\phi_<(x,t)} \mathrm{e}^{-\mathrm{i}\alpha\phi_>(0,0)} \mathrm{e}^{-\mathrm{i}\alpha\phi_<(0,0)} \rangle$$

$$= \langle \mathrm{e}^{\mathrm{i}\alpha\phi_<(x,t)} \mathrm{e}^{-\mathrm{i}\alpha\phi_>(0,0)} \rangle = \underbrace{\mathrm{e}^{[\alpha\phi_<(x,t), \alpha\phi_>(0,0)]}}_{= \mathrm{e}^{\alpha^2 \langle \phi(x,t)\phi(0,0)\rangle}} \underbrace{\langle \mathrm{e}^{-\mathrm{i}\alpha\phi_>(0,0)} \mathrm{e}^{\mathrm{i}\alpha\phi_<(x,t)} \rangle}_{=1}$$

$$= \begin{cases} (1 - \mathrm{e}^{\mathrm{i}2\pi \frac{x - vt + \mathrm{i}0^+}{L}})^{-\alpha^2/\kappa}, & t > 0 \\ (1 - \mathrm{e}^{-\mathrm{i}2\pi \frac{x - vt - \mathrm{i}0^+}{L}})^{-\alpha^2/\kappa}, & t < 0 \end{cases}$$

$$\approx \frac{(L/2\pi)^{\alpha^2/\kappa}}{[-\mathrm{i}(x - vt)\mathrm{sgn}(t) + 0^+]^{\alpha^2/\kappa}} = \frac{(L/2\pi)^{1/\kappa} \mathrm{e}^{\mathrm{i}\frac{1}{\kappa}\frac{\pi}{2}\mathrm{sgn}((x-vt)t)}}{|x - vt|^{\alpha^2/\kappa}}$$

The value of the mutivalued function is in the branch of $0^+ \to +\infty$.

Correlation function of $\mathrm{e}^{\mathrm{i}\theta}$ and no symmtery breaking

$$\langle \mathcal{T}[:\mathrm{e}^{\mathrm{i}\theta(x,t)}::\mathrm{e}^{-\mathrm{i}\theta(0,0)}:]\rangle \qquad \mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}(\alpha\phi_1+\alpha\phi_2)}, \alpha = (2V_1)^{-1/2}$$

$$= \langle \mathcal{T}[:\mathrm{e}^{\frac{\alpha}{2}\mathrm{i}\phi_1(x,t)}::\mathrm{e}^{-\frac{\alpha}{2}\mathrm{i}\phi_1(0,0)}:]\rangle\langle \mathcal{T}[:\mathrm{e}^{\frac{\alpha}{2}\mathrm{i}\phi_2(x,t)}::\mathrm{e}^{-\frac{\alpha}{2}\mathrm{i}\phi_2(0,0)}:]\rangle$$

$$= \begin{cases} (1-\mathrm{e}^{\mathrm{i}2\pi\frac{-x-vt+\mathrm{i}0^+}{L}})^{-\alpha^2/4\kappa}(1-\mathrm{e}^{\mathrm{i}2\pi\frac{x-vt+\mathrm{i}0^+}{L}})^{-\alpha^2/4\kappa}, & t>0 \\ (1-\mathrm{e}^{-\mathrm{i}2\pi\frac{-x-vt-\mathrm{i}0^+}{L}})^{-\alpha/4\kappa}(1-\mathrm{e}^{-\mathrm{i}2\pi\frac{x-vt-\mathrm{i}0^+}{L}})^{-\alpha/4\kappa}, & t<0 \end{cases}$$

$$= \frac{(L/2\pi)^{\alpha^2/2\kappa}}{[-\mathrm{i}(x-vt)\mathrm{sgn}(t)+0^+]^{\alpha^2/4\kappa}[-\mathrm{i}(-x-vt)\mathrm{sgn}(t)+0^+]^{\alpha/4\kappa}}$$

$$= \frac{(L/2\pi)^{2\gamma}}{(x^2-v^2t^2+\mathrm{i}2vt\mathrm{sgn}(t)0^+ + (0^+)^2)^\gamma} = \frac{(L/2\pi)^{2\gamma}}{(x^2-v^2t^2+\mathrm{i}0^+)^\gamma}$$

$\gamma = \lambda^2/4\kappa = \sqrt{V_1V_2}/2V_1$ (choose the positive branch for $x \to \infty$).

- Imaginary-time ($\tau = \mathrm{i}t$) correlation is simplified $\frac{(L/2\pi)^{2\gamma}}{(z\bar{z})^\gamma}$, $z = x + \mathrm{i}v\tau$

- 1d supperfluid (boson condensation or $U(1)$ symm. breaking) only has an **algebraic long range order**, not real **long range order** (since $\langle :\mathrm{e}^{\mathrm{i}\theta(x,0)}::\mathrm{e}^{-\mathrm{i}\theta(0,0)}:\rangle|_{x\to\infty} \not\to \mathit{const.}$) **Conitinous symmetry cannot spontaneously broken in 1D. It can only "nearly broken"**

Correlation function and spectral function of $\mathrm{e}^{\mathrm{i}\theta} \sim \sigma^+$

$$G(x,t) = \mathrm{i}\langle T[:\mathrm{e}^{\mathrm{i}\theta(x,t)}::\mathrm{e}^{-\mathrm{i}\theta(0,0)}:]\rangle$$

$$= \mathrm{i}(1-\mathrm{e}^{\mathrm{i}2\pi\frac{x-vt}{L}\mathrm{sgn}(t)})^{-\gamma}(1-\mathrm{e}^{\mathrm{i}2\pi\frac{-x-vt}{L}\mathrm{sgn}(t)})^{-\gamma}$$

$$= \sum_n C_{m,n}\mathrm{i}\,\mathrm{e}^{\mathrm{i}(m\frac{2\pi}{L}x-n\frac{2\pi v}{L}t)\mathrm{sgn}(t)} = \sum_n C_{m,n}\mathrm{i}\,\mathrm{e}^{\mathrm{i}(\kappa_m x-E_n t)\mathrm{sgn}(t)}$$

$$I(k,\omega) = \sum_n C_{m,n}[\delta(k-\kappa_m)\delta(\omega-E_n)+\delta(k+\kappa_m)\delta(\omega+E_n)]$$

Fourier transformation of $G(x,t)$:

$$\int_0^L \mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}t\ \mathrm{e}^{-\mathrm{i}(kx-\omega t)}\mathrm{i}\,\mathrm{e}^{\mathrm{i}(\kappa_m x-E_n t)\mathrm{sgn}(t)}$$

$$= \int_0^L \mathrm{d}x\int_0^{\infty}\mathrm{d}t\ \mathrm{e}^{-\mathrm{i}[kx-(\omega+\mathrm{i}0^+)t]}\mathrm{i}\,\mathrm{e}^{\mathrm{i}(\kappa_m x-E_n t)} + (t<0)$$

$$= \underbrace{\delta(k-\kappa_m)}_{L\delta_{k,\kappa_m}}\frac{\mathrm{i}}{-\mathrm{i}(\omega-E_n+\mathrm{i}0^+)} = \underbrace{\delta(k-\kappa_m)}_{L\delta_{k,\kappa_m}}[\frac{-1}{\omega-E_n}+\mathrm{i}\pi\delta(\omega-E_n)]$$

$$I(k,\omega) = \mathrm{Im}\,G(k,\omega)/\pi$$

Correlation function and spectral function of $e^{i\theta} \sim \sigma^+$

Correlation function of $e^{i\theta} \sim \sigma^+$

$$G(x.t) = \frac{i(L/2\pi)^{2\gamma}}{(x^2 - v^2t^2 + i0^+)^\gamma} = \frac{i(L/2\pi)^{2\gamma}}{(y_1y_2 + i0^+)^\gamma}$$

where $y_1 = x + vt,\ y_2 = x - vt$. We find

$$\begin{aligned} G(k,\omega) &= \int dx\,dt\ e^{-i(kx-\omega t)} \frac{i(L/2\pi)^{2\gamma}}{(x^2 - v^2t^2 + i0^+)^\gamma} \\ &= \int dx\,dt\ e^{-i\frac{1}{2}[k(y_1+y_2)-v^{-1}\omega(y_1-y_2)]} \frac{i(L/2\pi)^{2\gamma}}{(y_1y_2 + i0^+)^\gamma} \\ &\sim \int dy_1\,dy_2\ \frac{i\,e^{-i\frac{1}{2}[(k-\frac{\omega}{v})y_1+(k+\frac{\omega}{v})y_2]}}{(y_1y_2 + i0^+)^\gamma} \end{aligned}$$

up to a positive factor.

When taking the fractional power γ, choose the possitive brach for $y_1y_2 > 0$. For $y_1y_2 > 0$, choose branch that connect to the possitive brach for $y_1y_2 > 0$. Now the term $i0^+$ becomes important.

- $y_1 > 0,\ y_2 > 0$: Using $\int_0^\infty \mathrm{d}x \frac{\mathrm{e}^{-ax}}{x^\alpha} = \Gamma(1-\alpha)a^{\alpha-1},\ \mathrm{Re}(a) > 0$ and inserting 0^+ to make sure $\mathrm{Re}(a) > 0$, we find

$$
\begin{aligned}
G_{++}(k,\omega) &= \mathrm{i}\int_0^\infty \mathrm{d}y_1 \int_0^\infty \mathrm{d}y_2\ \frac{\mathrm{e}^{-\mathrm{i}\,\frac{1}{2}(k-\frac{\omega}{v}-\mathrm{i}0^+)y_1}\,\mathrm{e}^{-\mathrm{i}\,\frac{1}{2}(k+\frac{\omega}{v}-\mathrm{i}0^+)y_2}}{(y_1y_2+\mathrm{i}0^+)^\gamma} \\
&= \mathrm{i}\Big(\frac{\mathrm{i}(k-\frac{\omega}{v})+0^+}{2}\Big)^{\gamma-1}\Gamma(1-\gamma)\Big(\frac{\mathrm{i}(k+\frac{\omega}{v})+0^+}{2}\Big)^{\gamma-1}\Gamma(1-\gamma) \\
&= \mathrm{i}\,\mathrm{e}^{\mathrm{i}\,\frac{\pi}{2}(\gamma-1)[\mathrm{sgn}(vk-\omega)+\mathrm{sgn}(vk+\omega)]} \\
&\quad \Big(\frac{|vk-\omega|}{2v}\Big)^{\gamma-1}\Big(\frac{|vk+\omega|}{2v}\Big)^{\gamma-1}\Gamma^2(1-\gamma)
\end{aligned}
$$

Correlation function and spectral function of $\mathrm{e}^{\mathrm{i}\theta} \sim \sigma^+$

- $y_1 > 0,\ y_2 < 0$: Using $\int_0^\infty \mathrm{d}x \frac{\mathrm{e}^{-ax}}{x^\alpha} = \Gamma(1-\alpha)a^{\alpha-1},\ \mathrm{Re}(a) > 0$ and inserting 0^+ to make sure $\mathrm{Re}(a) > 0$, we find

$$G_{+-}(k,\omega) = \mathrm{i}\int_0^\infty \mathrm{d}y_1 \int_{-\infty}^0 \mathrm{d}y_2\ \frac{\mathrm{e}^{-\mathrm{i}\frac{1}{2}(k-\frac{\omega}{v}-\mathrm{i}0^+)y_1}\mathrm{e}^{-\mathrm{i}\frac{1}{2}(k+\frac{\omega}{v}+\mathrm{i}0^+)y_2}}{(y_1y_2+\mathrm{i}0^+)^\gamma}$$

$$= \mathrm{i}\int_0^\infty \mathrm{d}y_1 \int_0^\infty \mathrm{d}y_2\ \frac{\mathrm{e}^{-\mathrm{i}\frac{1}{2}(k-\frac{\omega}{v}-\mathrm{i}0^+)y_1}\mathrm{e}^{\mathrm{i}\frac{1}{2}(k+\frac{\omega}{v}+\mathrm{i}0^+)y_2}}{(-y_1y_2+\mathrm{i}0^+)^\gamma}$$

$$= \mathrm{i}\Big(\frac{\mathrm{i}(k-\frac{\omega}{v})+0^+}{2}\Big)^{\gamma-1}\Big(\frac{-\mathrm{i}(k+\frac{\omega}{v})+0^+}{2}\Big)^{\gamma-1}\mathrm{e}^{-\mathrm{i}\pi\gamma}\Gamma^2(1-\gamma)$$

$$= \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\pi\gamma}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\gamma-1)[\mathrm{sgn}(vk-\omega)-\mathrm{sgn}(vk+\omega)]}$$

$$\Big(\frac{|vk-\omega|}{2v}\Big)^{\gamma-1}\Big(\frac{|vk+\omega|}{2v}\Big)^{\gamma-1}\Gamma^2(1-\gamma)$$

- $y_1 < 0,\ y_2 > 0$: Using $\int_0^\infty dx \frac{e^{-ax}}{x^\alpha} = \Gamma(1-\alpha)a^{\alpha-1},\ \mathrm{Re}(a) > 0$ and inserting 0^+ to make sure $\mathrm{Re}(a) > 0$, we find

$$
\begin{aligned}
G_{-+}(k,\omega) &= i \int_{-\infty}^{0} dy_1 \int_0^\infty dy_2\, \frac{e^{-i\frac{1}{2}(k-\frac{\omega}{v}+i0^+)y_1}\, e^{-i\frac{1}{2}(k+\frac{\omega}{v}-i0^+)y_2}}{(y_1y_2+i0^+)^\gamma} \\
&= i \int_0^\infty dy_1 \int_0^\infty dy_2\, \frac{e^{i\frac{1}{2}(k-\frac{\omega}{v}+i0^+)y_1}\, e^{-i\frac{1}{2}(k+\frac{\omega}{v}-i0^+)y_2}}{(-y_1y_2+i0^+)^\gamma} \\
&= i \Big(\frac{-i(k-\frac{\omega}{v})+0^+}{2}\Big)^{\gamma-1} \Big(\frac{i(k+\frac{\omega}{v})+0^+}{2}\Big)^{\gamma-1} e^{-i\pi\gamma}\Gamma^2(1-\gamma) \\
&= i\, e^{-i\pi\gamma} e^{i\frac{\pi}{2}(\gamma-1)[-\mathrm{sgn}(vk-\omega)+\mathrm{sgn}(vk+\omega)]} \\
&\qquad \Big(\frac{|vk-\omega|}{2v}\Big)^{\gamma-1} \Big(\frac{|vk+\omega|}{2v}\Big)^{\gamma-1} \Gamma^2(1-\gamma)
\end{aligned}
$$

Correlation function and spectral function of $\mathrm{e}^{\mathrm{i}\theta} \sim \sigma^+$

- $y_1 < 0,\ y_2 < 0$: Using $\int_0^\infty \mathrm{d}x \frac{\mathrm{e}^{-ax}}{x^\alpha} = \Gamma(1-\alpha)a^{\alpha-1},\ \mathrm{Re}(a) > 0$ and inserting 0^+ to make sure $\mathrm{Re}(a) > 0$, we find

$$G_{--}(k,\omega) = \mathrm{i}\int_{-\infty}^{0} \mathrm{d}y_1 \int_{-\infty}^{0} \mathrm{d}y_2\ \frac{\mathrm{e}^{-\mathrm{i}\frac{1}{2}(k-\frac{\omega}{v}+\mathrm{i}0^+)y_1}\,\mathrm{e}^{-\mathrm{i}\frac{1}{2}(k+\frac{\omega}{v}+\mathrm{i}0^+)y_2}}{(y_1y_2+\mathrm{i}0^+)^\gamma}$$

$$= \mathrm{i}\int_{0}^{\infty} \mathrm{d}y_1 \int_{0}^{\infty} \mathrm{d}y_2\ \frac{\mathrm{e}^{\mathrm{i}\frac{1}{2}(k-\frac{\omega}{v}+\mathrm{i}0^+)y_1}\,\mathrm{e}^{\mathrm{i}\frac{1}{2}(k+\frac{\omega}{v}+\mathrm{i}0^+)y_2}}{(y_1y_2+\mathrm{i}0^+)^\gamma}$$

$$= \mathrm{i}\Big(\frac{-\mathrm{i}(k-\frac{\omega}{v})+0^+}{2}\Big)^{\gamma-1}\Big(\frac{-\mathrm{i}(k+\frac{\omega}{v})+0^+}{2}\Big)^{\gamma-1}\Gamma^2(1-\gamma)$$

$$= \mathrm{i}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\gamma-1)[-\mathrm{sgn}(vk-\omega)-\mathrm{sgn}(vk+\omega)]}$$

$$\Big(\frac{|vk-\omega|}{2v}\Big)^{\gamma-1}\Big(\frac{|vk+\omega|}{2v}\Big)^{\gamma-1}\Gamma^2(1-\gamma)$$

Correlation function and spectral function of $\mathrm{e}^{\mathrm{i}\theta} \sim \sigma^+$

$$G(k,\omega) \sim \mathrm{i}\Big(\frac{|vk-\omega|}{2v}\Big)^{\gamma-1}\Big(\frac{|vk+\omega|}{2v}\Big)^{\gamma-1}\Gamma^2(1-\gamma)\times$$

$$\Big(\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\gamma-1)[\mathrm{sgn}(vk-\omega)+\mathrm{sgn}(vk+\omega)]} + \mathrm{e}^{-\mathrm{i}\pi\gamma}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\gamma-1)[\mathrm{sgn}(vk-\omega)-\mathrm{sgn}(vk+\omega)]}$$

$$+ \mathrm{e}^{-\mathrm{i}\pi\gamma}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\gamma-1)[-\mathrm{sgn}(vk-\omega)+\mathrm{sgn}(vk+\omega)]} + \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\gamma-1)[-\mathrm{sgn}(vk-\omega)-\mathrm{sgn}(vk+\omega)]}\Big)$$

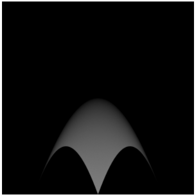
$$= \mathrm{i}\Big(\frac{|vk-\omega|}{2v}\Big)^{\gamma-1}\Big(\frac{|vk+\omega|}{2v}\Big)^{\gamma-1}\Gamma^2(1-\gamma)\times$$

$$\begin{cases} -\mathrm{e}^{\mathrm{i}\pi\gamma} + \mathrm{e}^{-\mathrm{i}\pi\gamma} + \mathrm{e}^{-\mathrm{i}\pi\gamma} - \mathrm{e}^{-\mathrm{i}\pi\gamma} = -2\mathrm{i}\sin(\pi\gamma), & vk-\omega>0, vk+\omega>0 \\ -\mathrm{e}^{-\mathrm{i}\pi\gamma} + \mathrm{e}^{-\mathrm{i}\pi\gamma} + \mathrm{e}^{-\mathrm{i}\pi\gamma} - \mathrm{e}^{\mathrm{i}\pi\gamma} = -2\mathrm{i}\sin(\pi\gamma), & vk-\omega<0, vk+\omega<0 \\ 1-1-\mathrm{e}^{-\mathrm{i}2\pi\gamma}+1 = 1-\mathrm{e}^{-\mathrm{i}2\pi\gamma}, & vk-\omega>0, vk+\omega<0 \\ 1-\mathrm{e}^{-\mathrm{i}2\pi\gamma}-1+1 = 1-\mathrm{e}^{-\mathrm{i}2\pi\gamma}, & vk-\omega<0, vk+\omega>0 \end{cases}$$

Spectral function: $I(k,\omega) = \Big(\frac{|vk-\omega|}{2v}\Big)^{\gamma-1}\Big(\frac{|vk+\omega|}{2v}\Big)^{\gamma-1}\Gamma^2(1-\gamma)\times$

$$\begin{cases} 0, & (\omega-vk)(\omega+vk)<0 \\ 1-\cos(2\pi\gamma), & (\omega-vk)(\omega+vk)>0 \end{cases}$$

$k = 0$ modes, and large momentum sectors



- Our theory so far contain only exications desbribed by oscilators a_k, $k = \frac{2\pi}{L} \times \text{int.}$.
- Our theory so far can produce exication near $k = 0$, but not near $k = k_b = 2\pi \frac{N}{L}$.
- The correlation $\langle \mathcal{T}[: \mathrm{e}^{\mathrm{i}\theta(x,t)} :: \mathrm{e}^{\mathrm{i}\theta(0,0)} :] \rangle \sim (x^2 - v^2 t^2)^{-1/4\kappa} + 0\mathrm{e}^{\mathrm{i}k_b x}$ contains nothing near k_b.

- To inlcude the low energy sectors with large momentum, we need to include $k = 0$ modes:

$$\text{Low energy excitations} = (k \neq 0 \text{ modes}) \otimes (k = 0 \text{ modes})$$

- Consider θ, φ non-linear σ-model:

$$L = \int \mathrm{d}x \; (\frac{1}{2\pi}\partial_x \varphi + \frac{\bar{\phi}^2}{a})\partial_t \theta - \frac{v}{4\pi}(\partial_x \theta)^2 - \frac{v}{4\pi}(\partial_x \varphi)^2$$

- The $k = 0$ sectors are labeled by $w_\theta, w_\varphi \in \mathbb{Z}$ (Only $q = \partial\varphi$ is physical):

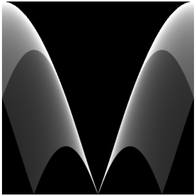
$\theta(x) = w_\theta \frac{2\pi}{L} x + \theta_0 + (k \neq 0 \text{ modes})$, $\quad \varphi(x) = w_\varphi \frac{2\pi}{L} x + (k \neq 0 \text{ modes})$.

$L = (w_\varphi + \frac{\bar{\phi}^2 L}{a})\dot{\theta}_0 - \frac{1}{2}\frac{2\pi}{L} v (w_\theta^2 + w_\varphi^2) \;\to E = \frac{1}{2}\frac{2\pi}{L} v (w_\theta^2 + w_\varphi^2)$

The physical meanings of winding numbers w_θ, w_φ from the connection to the lattice model

- **What is the meaning of w_φ (angular momentum of θ_0)?**
  We note that $2\bar\phi a^{-1} q = \kappa \partial_x\varphi/\pi = \partial_x \varphi/2\pi = w_\varphi/L$.
  So $w_\varphi = \int \mathrm{d}x\ 2\bar\phi a^{-1} q = \sum_i 2\bar\phi q_i$

  Spectral function of n_i

  But what is $\sum_i 2\bar\phi q_i$? Remember that $\phi_i = \bar\phi + q_i$
  and $|\phi_i\rangle = \frac{|\uparrow\rangle + \phi_i|\downarrow\rangle}{\sqrt{1+|\phi_i|^2}} = \frac{|0\rangle + \phi_i|1\rangle}{\sqrt{1+|\phi_i|^2}}$.
  So $\langle n_i\rangle = \frac{|\phi_i|^2}{1+|\phi_i|^2} \approx |\phi_i|^2 \approx \bar\phi^2 + 2\bar\phi q_i$
  Thus the canonical momentum of θ_0,
  $\frac{\bar\phi^2 L}{a} + w_\varphi = \sum_i(\bar\phi^2 + 2\bar\phi q_i) = \sum_i n_i = N$, is the total number of the bosons. This should be an exact result, since $\theta_0 \sim \theta_0 + 2\pi$ and its anluar momenta are quantized as integers.

- **What is the meaning of w_θ?**
  A non-zero w_θ gives rise $\phi_i = \bar\phi \mathrm{e}^{\mathrm{i}\, w_\theta x \frac{2\pi}{L}}$. Each boson carries momentum $w_\theta \frac{2\pi}{L}$. The total momentum is $w_\theta \frac{2\pi N_0}{L} = w_\theta k_b$.

Obtain the meanings of $w_\theta,\ w_\varphi$ within the field theory

$$L=\int \mathrm{d}x\ (\frac{1}{2\pi}\partial_x\varphi+\frac{\bar\phi^2}{a})\partial_t\theta-\frac{v}{4\pi}(\partial_x\theta)^2-\frac{v}{4\pi}(\partial_x\varphi)^2$$

- The $U(1)$ symmetry transformation is given by $\theta\to\theta+\theta_0$. The angular momentum of θ_0 is the total number of the $U(1)$ charges (*ie* the number of bosons). From the corrsponding Lagrangian $L=(w_\phi+\frac{\bar\phi^2 L}{a})\dot\theta_0+\cdots$, we see the $U(1)$ charge is $Q=w_\phi-\frac{\bar\phi^2 L}{a}$
- The translation symmetry transformation is given by $\theta(x)\to\theta(x-x_0),\ \varphi(x)\to\varphi(x-x_0)$. The cannonical momentum of x_0 is the total momentum.

\- We consider the field of form $\theta(x-x_0),\varphi(x-x_0)$ and only x_0 is dynamical, *ie* time denpendent (the $k\neq 0$ mode can be dropped):

$$\theta(x,t)=w_\theta\frac{2\pi}{L}(x+x_0(t))+\theta_0+(k\neq 0\text{ modes}),$$
$$\varphi(x,t)=w_\varphi\frac{2\pi}{L}(x+x_0(t))+(k\neq 0\text{ modes}).$$

From the corresponding Lagrangian $L=(w_\phi+\frac{\bar\phi^2 L}{a})\frac{2\pi}{L}w_\theta\dot x_0+\cdots$, we see the total momentum is $K=N\frac{2\pi}{L}w_\theta=k_b w_\theta$.

Winding-number changing operators

$$L = \int \mathrm{d}x \, (\frac{1}{2\pi}\partial_x\varphi + \frac{\bar{\phi}^2 L}{a})\partial_t\theta - \frac{v}{4\pi}(\partial_x\theta)^2 - \frac{v}{4\pi}(\partial_x\varphi)^2$$

- The **local operator** $\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}\alpha(\phi_1+\phi_2)}$ changes the particle number N by -1, *ie* change the winding number of φ, w_φ, by -1.
- To see this explicitly $[\theta(x), \frac{1}{2\pi}\partial_y\varphi(y)] = \mathrm{i}\delta(x-y)$
  We find $[\theta(x), \Delta\varphi] = \mathrm{i}2\pi$ where $\Delta\varphi = \varphi(+\infty) - \varphi(-\infty)$.
  Thus $\theta(x) = \mathrm{i}2\pi\frac{\mathrm{d}}{\mathrm{d}\Delta\varphi} +$ commutants of $\Delta\varphi$, and $\mathrm{e}^{\mathrm{i}\theta(x)} = \mathrm{e}^{-2\pi\frac{\mathrm{d}}{\mathrm{d}\Delta\varphi}+\cdots}$ is an operator that changes $\Delta\varphi$ by -2π, or w_φ by -1, or particle number by -1
- Similarly, we have $[\theta(x), \varphi(y)] = -\mathrm{i}2\pi\Theta(x-y)$
  $\to [\partial_x\theta(x), \varphi(y)] = -\mathrm{i}2\pi\delta(x-y)$
  We find $[\Delta\theta, \varphi(y)] = -\mathrm{i}2\pi$ where $\Delta\theta = \theta(+\infty) - \theta(-\infty)$.
  Thus $\varphi(y) = \mathrm{i}2\pi\frac{\mathrm{d}}{\mathrm{d}\Delta\theta}$, and $\mathrm{e}^{\mathrm{i}\varphi(x)} = \mathrm{e}^{-2\pi\frac{\mathrm{d}}{\mathrm{d}\Delta\theta}}$ is an operator that changes $\Delta\theta$ by -2π, or change w_θ by -1 (*ie* total momentum by $-k_b$).

Local operators in 1D XY-model (superfluid)

- Lattice operators

$$\sigma_i^z = (\#\partial_x\theta + \#\partial_x\varphi) + \#\mathrm{e}^{-\mathrm{i}k_b x}\mathrm{e}^{\mathrm{i}\varphi(x)} + \cdots$$
$$\sigma_i^+ = (\# + \#\partial_x\theta + \#\partial_x\varphi)\mathrm{e}^{-\mathrm{i}\theta(x)} + \#\mathrm{e}^{-\mathrm{i}k_b x}\mathrm{e}^{-\mathrm{i}\theta(x)}\mathrm{e}^{\mathrm{i}\varphi(x)} + \cdots$$

- Set of local operators: $\partial_x\theta,\ \partial_x\varphi,\ \underbrace{\mathrm{e}^{\mathrm{i}(m_\theta\theta+m_\varphi\varphi)}}_{\text{change sectors}}$

or (from $\theta = \alpha(\phi_1+\phi_2)$, $\varphi = \beta(\phi_1-\phi_2)$)

$$\partial_x\phi_1,\ \partial_x\phi_2,\ \underbrace{\mathrm{e}^{\mathrm{i}(m_1\phi_1+m_2\phi_2)}}_{\text{change sectors}}$$

where $m_1 = \alpha m_\theta + \beta m_\varphi,\quad m_2 = \alpha m_\theta - \beta m_\varphi.$

- **Fractionalization in XY-model (superfluid)**

A boson creation operator $\sigma^+ \sim \mathrm{e}^{\mathrm{i}\theta}$ (spin flip operator $\Delta S^z = 1$)

$$\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}\alpha(\phi_1+\phi_2)},\quad \phi_1 \text{ left-mover},\quad \phi_2 \text{ right-mover}$$

$\mathrm{e}^{\mathrm{i}\alpha\phi_2}$ creats half boson (spin-1/2) in right-moving sector
$\mathrm{e}^{\mathrm{i}\alpha\phi_1}$ creats half boson (spin-1/2) in left-moving sector

Lattice translation and $U(1)$ symm. are not independent

- For a 1d superfluid of per-site-density n_b the ground state is described by a field $\phi(x) = \bar{\phi} e^{-i\theta(x)}, \theta(x) = 0$. The total momentum of the ground state is $K = 0$.
- We do a $U(1)$ symmetry twist: $\theta(L) = \theta(0) \to \theta(L) = \theta(0) + \Delta\theta$. The twisted state is described by a field $\theta(x) = \frac{\Delta\theta}{L} x$. The total momentum of the twisted state is $K = k_b \frac{\Delta\theta}{2\pi} = N + \Delta k = N \frac{\Delta\theta}{L}$.

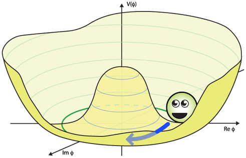
- $U(1)$ symmetry twist = momentum bost $k_i \to k_i + \frac{\Delta\theta}{L}$.
  Doing a symmetry twist operation in a symmetry can change the quantum number of another symmetry $\to$ mixed anomaly
- A 2π $U(1)$ symmetry twist can change the total crystal momentum by $k_b = 2\pi n_b$. Since 2π-crystal-momentum = 0-crystal-momentum, our bosonic system have an mixed translation-$U(1)$ anomaly when boson number per site $n_b \notin \mathbb{Z}$. $\to$ **There is no translation and $U(1)$ symmetric product state.**

- We do a translation symmetry twist operation by adding ΔL sites $\to$ change the total boson numbers (the $U(1)$ charges) of system by $n_b \Delta L$.

1d field theory – non-linear σ-model

- "Coordinate space" Lagrangian (rotor model): subsitute one of the EOM $\frac{1}{2\pi}\partial_t\theta = \frac{1}{2\pi}V_2\partial_x\varphi$ into the phase space Lagrangian

$$L = \int \mathrm{d}x\ \frac{V_2^{-1}}{4\pi}(\partial_t\theta)^2 - \frac{V_1}{4\pi}(\partial_x\theta)^2 \ \underbrace{+\frac{\bar\phi^2}{a}\partial_t\theta}_{\text{a topo. term}}$$



$$= \int \mathrm{d}x\ \frac{V_2^{-1}}{4\pi}(\mathrm{i}\,u^*\partial_t u)^2 - \frac{V_1}{4\pi}(\mathrm{i}\,u^*\partial_x u)^2 - \mathrm{i}\,\frac{\bar\phi^2}{a}u^*\partial_t u$$

- The field is really $u = \mathrm{e}^{\mathrm{i}\theta}$, not θ. The above is the so called non-linear σ-model, where the field is a map from space-time manifold to the **target space** S^1: $M^{d+1}_{\text{space-time}} \to U(1)$.
- In general, the target space is the symmetric space $G_{\text{symm}}/G_{\text{unbroken}}$ (the minima of the symmetry breaking potential).
- The topological term $\mathrm{i}\,\frac{\bar\phi^2}{a}u^*\partial_t u$ cannot be dropped (since it is not a total derivative). When $\bar\phi^2 = n \notin \mathbb{Z}$, the topological term makes it impossible for the non-linear σ-model to have a gapped phase (an effect of mixed anomaly between $U(1)$ symmetry and tranlation symmetry).

- The above is a **low energy effective theory** for $U(1)$ symm breaking

Symmetry, gauging, and conservation

- Consider a system described by a complex field u

$$S = \int \mathrm{d}t\mathrm{d}x\mathcal{L}(u)$$

with $U(1)$ symmetry: $\mathcal{L}(\mathrm{e}^{\mathrm{i}\lambda}u) = \mathcal{L}(u)$. We like to show that the system has an conserved current j^μ, $\mu = t, x$: $\partial_t j^t + \partial_x j^x = \partial_\mu j^\mu = 0$.

- **Gauge the $U(1)$ symmetry**:

- $u(x) \to \mathrm{e}^{\mathrm{i}\lambda_I(x)}u(x)$ gives rise to $u_I^*\partial_\mu u_I \to u_I^*(\partial_\mu + \mathrm{i}\partial_\mu\lambda_I)u_I$, $\mu = t, x$.
- Replacing $\partial_\mu\lambda_I$ by a vector potential A_μ^I: $u_I^*(\partial_\mu + \mathrm{i}A_\mu^I)u_I$ gives rise to a gauged theory $\mathcal{L} \to \mathcal{L}(u, A_\mu)$. Here A_μ is viewed as non-dynamical background field. We have

$$\mathcal{L}(u, A_\mu) = \mathcal{L}(\mathrm{e}^{\mathrm{i}\lambda}u, A_\mu - \partial_\mu\lambda)$$

- The $U(1)$ current of the gauged theory (setting $A_\mu = 0$ gives rise to the $U(1)$ current of the original theory)

$$\delta S = \int \mathrm{d}t\mathrm{d}x\, j^\mu\delta A_\mu, \quad j^\mu = \frac{\delta\mathcal{L}(u, A_\mu)}{\delta A_\mu}.$$

Symmetry, gauging, and conservation

- The current conservation:
$$\delta S = \int \mathrm{d}^2x^\mu \ \mathcal{L}(\mathrm{e}^{\mathrm{i}\lambda}u, A_\mu) - \mathcal{L}(u, A_\mu)$$
$$= \int \mathrm{d}^2x^\mu \, \mathcal{L}(u, A_\mu + \partial_\mu\lambda) - \mathcal{L}(u, A_\mu) = \int \mathrm{d}^2x^\mu \, j^\mu \partial_\mu \lambda = -\int \mathrm{d}^2x^\mu \, \lambda \partial_\mu j^\mu$$
If $u(x,t)$ satisfies the equation of motion, then the cooresponding $\delta S = 0$. This allows us to show the existance of a conserved current
$$\partial_\mu j^\mu(u) = 0.$$

- **Example**: $\partial_\mu\theta = -\mathrm{i}\, u^*\partial_\mu u \to \partial_\mu\theta + A_\mu = -\mathrm{i}\, u^*(\partial_\mu + \mathrm{i}A_\mu)u$
$$\mathcal{L} = \frac{V_2^{-1}}{4\pi}(\partial_t\theta)^2 - \frac{V_1}{4\pi}(\partial_x\theta)^2 + \frac{\bar\phi^2}{a}\partial_t\theta$$
$$\to \ \mathcal{L} = \ \frac{V_2^{-1}}{4\pi}(\partial_t\theta + A_t)^2 - \frac{V_1}{4\pi}(\partial_x\theta + A_x)^2 + \frac{\bar\phi^2}{a}(\partial_t\theta + A_t)$$
$$\to \ \ j^\mu = \frac{\delta\mathcal{L}}{\delta A_\mu}, \ \ j^t = \frac{V_2^{-1}}{2\pi}(\partial_t\theta + A_t), \ j^x = -\frac{V_1}{2\pi}(\partial_x\theta + A_x).$$

Another example of gauging symmetry

Consider the following effective theory for 1d bosonic superfluid

$$L = \int \mathrm{d}x \; \frac{K_{IJ}}{4\pi}\partial_x\varphi_I\partial_t\varphi_J - \frac{V_{IJ}}{4\pi}\partial_x\varphi_I\partial_x\varphi_J + q_I\partial\phi_I$$
$$= \int \mathrm{d}x \; \frac{K_{IJ}}{4\pi}\partial_x u_I^*\partial_t u_J - \frac{V_{IJ}}{4\pi}\partial_x u_I^*\partial_x u_J - \mathrm{i}\, q_I u_I^*\partial_t u_I$$

$$I, J = 1, 2, \quad \varphi_I \sim \varphi_I + 2\pi, \quad u_I = \mathrm{e}^{\mathrm{i}\varphi_I}, \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} \frac{\bar\phi^2}{a} \\ 0 \end{pmatrix}.$$

- The effective field theory has two $U(1)$ symmetries:

- $\varphi_1 \to \varphi_1 + \lambda_1$ for boson number conservation
  Conjuate of λ_1 is $\int \mathrm{d}x \, \frac{1}{2\pi}\partial_x\varphi_2 = w_\varphi = N$.
- $\varphi_2 \to \varphi_2 + \lambda_2$ for momentum conservation.
  Conjuate of λ_2 is $\int \mathrm{d}x \, \frac{1}{2\pi}\partial_x\varphi_1 = w_\theta = K/k_b$.

Another example of gauging symmetry

- Gauging the two $U(1)$ symmetries:

- $u_I(x) \to \mathrm{e}^{\mathrm{i}\lambda_I(x)} u_I(x)$ gives rise to $u_I^* \partial_\mu u_I \to u_I^*(\partial_\mu + \mathrm{i}\partial_\mu \lambda_I) u_I$, $\mu = t, x$.
- Replacing $\partial_\mu \lambda_I$ by a vector potential A_μ^I gives rise to a gauged theory

$$\mathcal{L} = \frac{K_{IJ}}{4\pi}(\partial_x - \mathrm{i}A_x^I)u_I^*(\partial_t + \mathrm{i}A_t^J)u_J - \frac{V_{IJ}}{4\pi}(\partial_x - \mathrm{i}A_x^I)u_I^*(\partial_x + \mathrm{i}A_x^J)u_J$$
$$- \mathrm{i}\,q_I u_I^*(\partial_t + \mathrm{i}A_t^I)u_I$$
$$= \frac{K_{IJ}}{4\pi}(\partial_x\varphi_I + A_x^I)(\partial_t\varphi_J + A_t^J) - \frac{V_{IJ}}{4\pi}(\partial_x\varphi_I + A_x^I)(\partial_x\varphi_J + A_x^J) + q_I(\partial_t\varphi_I + A_t^I)$$

- Conserved current

$$j_I^t = \frac{K_{IJ}}{4\pi}(\partial_x\varphi_J + A_x^J) + q_I, \quad j_I^x = \frac{K_{IJ}}{4\pi}(\partial_t\varphi_J + A_t^J) - \frac{V_{IJ}}{2\pi}(\partial_x\varphi_J + A_x^J)$$

- Equaton of motion $\to$ conservation

$$-\frac{K_{IJ}}{4\pi}\partial_x(\partial_t\varphi_J + A_t^J) - \frac{K_{IJ}}{4\pi}\partial_t(\partial_x\varphi_J + A_x^J) + \frac{V_{IJ}}{2\pi}\partial_x(\partial_x\varphi_J + A_x^J) = 0$$
$$\to -\partial_t j_I^t - \partial_x j_I^x = 0$$

Symmetry twist, pumping, and anomaly

- But for certain background field $A^I_\mu(x,t)$, the equation of motion cannot be satisfied $\to$ non-conservation. **Symmetry twist** $\to$ **Pumping**
  Background field $A^I_\mu(x,t)$ = symmetry twist.
  Non-conservation = pumping

- Consider $A^I_t = 0$, A^I_x independent of x, but dependent on t. Equation of motion becomes

$$-\frac{K_{IJ}}{2\pi}\partial_x\partial_t\varphi_J + \frac{V_{IJ}}{2\pi}\partial_x^2\varphi_J = \frac{K_{IJ}}{4\pi}\partial_t A^J_x$$

It has no solution since, on a ring of size L,

$$0 = \int_0^L \mathrm{d}x\,\Big[-\frac{K_{IJ}}{2\pi}\partial_x\partial_t\varphi_J + \frac{V_{IJ}}{2\pi}\partial_x^2\varphi_J\Big] = \int_0^L \mathrm{d}x\,\frac{K_{IJ}}{4\pi}\partial_t A^J_x \neq 0$$

- The non-zero pumped $U(1)$ charge $\to$ $U(1)$ anomaly

$$\dot{Q}_I = \int_0^L \mathrm{d}x\,\partial_t j^t_I = \int_0^L \mathrm{d}x\,\partial_t\Big[\frac{K_{IJ}}{4\pi}(\partial_x\varphi_J + A^J_x) + q_I\Big] = \int_0^L \mathrm{d}x\,\partial_t\frac{K_{IJ}}{4\pi}A^J_x$$

Anomaly and mixed anomaly

Consider chiral boson theory

$$L = \int \mathrm{d}x \; \frac{K_{IJ}}{4\pi}\partial_x\varphi_I\partial_t\varphi_J - \frac{V_{IJ}}{4\pi}\partial_x\varphi_I\partial_x\varphi_J + q_I\partial\phi_I$$

$$\dot{Q}_I = \int_0^L \mathrm{d}x\,\partial_t j_I^t = \int_0^L \mathrm{d}x\,\partial_t \frac{K_{IJ}}{4\pi} A_x^J$$

- $K = (1)$, the theory is actually fermionic and describes a chiral fermion.

- The $U(1)$ symmetry twist pumps the $U(1)$ charge $\to$ $U(1)$ amonaly

- $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the theory is non-chiral describing 1d bosonic superfluid.

- The first $U(1)$ symmetry twist does not pump the first $U(1)$ charge. The first $U(1)$ is not anomalous.
- The second $U(1)$ symmetry twist does not pump the second $U(1)$ charge. The second $U(1)$ is not anomalous.
- The first $U(1)$ symmetry twist pumps the second $U(1)$ charge. The $U(1) \times U(1)$ symmetry has a mixed anomaly.

Anomaly and mixed anomaly

- $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the theory is non-chiral describing 1d Fermi liquid.

- The first $U(1)$ symmetry twist pumps the first $U(1)$ charge. The first $U(1)$ is anomalous.
- The second $U(1)$ symmetry twist pumps the second $U(1)$ charge. The second $U(1)$ is anomalous.

The "+" $U(1)$: $\varphi_1 \to \varphi_1 + \lambda_+$, $\varphi_2 \to \varphi_2 + \lambda_+$ $\to$ the fermion number
The "−" $U(1)$: $\varphi_1 \to \varphi_1 + \lambda_-$, $\varphi_2 \to \varphi_2 - \lambda_-$ $\to$ the total momentum provided that the fermion density is not zero.

- The "+" $U(1)$ symmetry twist does not pump the "+" $U(1)$ charge. The "+" $U(1)$ is not anomalous.
- The "−" $U(1)$ symmetry twist does not pump the "−" $U(1)$ charge. The "−" $U(1)$ is not anomalous.
- The "+" $U(1)$ symmetry twist does not pump the "−" $U(1)$ charge. There is a mixed anomaly between "+" $U(1)$ and "−" $U(1)$ symmetries. **The $U^2(1)$ symmetric state must be gapless**.

Why $K = (1)$ chiral boson theory describes chiral fermions

$K = (1)$ chiral boson field theory:

$$
\begin{aligned}
L &= \int \mathrm{d}x \, \frac{1}{4\pi} \partial_x \varphi \partial_t \varphi - \frac{V}{4\pi} \partial_x \varphi \partial_x \varphi \\
&= \sum_{k=-\infty}^{+\infty} \frac{-\mathrm{i}}{4\pi} k \varphi_{-k} \dot{\varphi}_k - \frac{V}{4\pi} k^2 \varphi_{-k} \varphi_k, \quad \varphi(x) = \sum_{k=-\infty}^{+\infty} \frac{\mathrm{e}^{\mathrm{i}\, kx}}{\sqrt{L}} \varphi_k \\
&= \sum_{k>0} \frac{-\mathrm{i}}{2\pi} k \varphi_{-k} \dot{\varphi}_k - \frac{V}{2\pi} k^2 \varphi_{-k} \varphi_k
\end{aligned}
$$

The canonical conjugate of φ is $\frac{1}{4\pi} \partial_y \varphi(y)$ or $\frac{1}{2\pi} \partial_y \varphi(y)$

$$
\begin{aligned}
&[\varphi_k, \frac{-\mathrm{i}\, k'}{2\pi} \varphi_{-k'}] = \mathrm{i} \delta_{k-k'}, \\
&[\varphi(x), \frac{1}{2\pi} \partial_y \varphi(y)] = \mathrm{i} \sum_k L^{-1} \mathrm{e}^{\mathrm{i}\, k(x-y)} = \mathrm{i} \int \frac{\mathrm{d}k}{2\pi} \mathrm{e}^{\mathrm{i}\, k(x-y)} \\
&[\varphi(x), \frac{1}{2\pi} \partial_y \varphi(y)] = \mathrm{i} \delta(x-y), \quad [\varphi(x), \varphi(y)] = \mathrm{i} \pi \mathrm{sgn}(x-y).
\end{aligned}
$$

Why $K = (1)$ chiral boson theory describes chiral fermions

- $\varphi(x)$ is a compcat field $\varphi(x) \sim \varphi(x) + 2\pi$. Thus $\varphi(x)$ is not an allowed operator. $\mathrm{e}^{\pm \mathrm{i}\varphi(x)}$ are allowed operators, all other allowed operators are generated by $\mathrm{e}^{\pm \mathrm{i}\varphi(x)}$.
- The allowed operators are non-local and should be forbiden:

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\varphi(x)}\mathrm{e}^{\mathrm{i}\varphi(y)} &= \mathrm{e}^{[\mathrm{i}\varphi(x),\mathrm{i}\varphi(y)]}\mathrm{e}^{\mathrm{i}\varphi(y)}\mathrm{e}^{\mathrm{i}\varphi(x)} \\ &= \mathrm{e}^{\mathrm{i}\pi \mathrm{sgn}(x-y)}\mathrm{e}^{\mathrm{i}\varphi(y)}\mathrm{e}^{\mathrm{i}\varphi(x)} = -\mathrm{e}^{\mathrm{i}\varphi(y)}\mathrm{e}^{\mathrm{i}\varphi(x)}\end{aligned}$$

- Or we regard the non-local operators $\mathrm{e}^{\pm \mathrm{i}\varphi(x)}$ as local fermion operator, and regard the chiral boson theory as a theroy for fermions.

\- The imaginary-time (time-ordered) correlation function for $\mathrm{e}^{\pm \mathrm{i}\varphi(x)}$:

$$\langle \mathrm{e}^{-\mathrm{i}\varphi(x,\tau)}\mathrm{e}^{\mathrm{i}\varphi(0)}\rangle \sim \frac{1}{x + \mathrm{i}v\tau} = \frac{1}{z}$$

which is identical to the correlation function of free chiral fermion $c(x,t)$, and allows us to identify $c(x,t) \sim \mathrm{e}^{\mathrm{i}\varphi(x,t)}$.

$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ boson theory describes 1d Fermi liquid

- **Bosonization**:

$$L = \int \mathrm{d}x\, \frac{1}{4\pi}\partial_x\varphi_R\partial_t\varphi_R - \frac{v_F}{4\pi}\partial_x\varphi_R\partial_x\varphi_R - \frac{1}{4\pi}\partial_x\varphi_L\partial_t\varphi_L - \frac{v_F}{4\pi}\partial_x\varphi_L\partial_x\varphi_L + q\partial_t(\varphi_R + \varphi_L)$$

describes 1d non-interacting fermions with Fermi velocity k_F.

- The fermion number $U(1)$ symmetry: $\varphi_R \to \varphi_R + \theta$, $\varphi_L \to \varphi_L + \theta$.
The canonical conjugate of θ is the fermion number $\to$ Fermion number density is given by $n_F = \frac{1}{2\pi}(\partial_x\varphi_R - \partial_x\varphi_L)$.

- Interacting 1d fermions via bosonization:

$$L = \int \mathrm{d}x\, \frac{1}{4\pi}\partial_x\varphi_R\partial_t\varphi_R - \frac{v_F}{4\pi}\partial_x\varphi_R\partial_x\varphi_R - \frac{1}{4\pi}\partial_x\varphi_L\partial_t\varphi_L - \frac{v_F}{4\pi}\partial_x\varphi_L\partial_x\varphi_L + \frac{V}{(2\pi)^2}(\partial_x\varphi_R - \partial_x\varphi_L)^2 + q\partial_t(\varphi_R + \varphi_L)$$

describes 1d interacting fermions, which allow us to compute fermion correlation $\langle c(x,t)c^\dagger(0)\rangle$, *etc* .

Fractionalization in general 1d chiral boson theory

$$L = \int \mathrm{d}x \; \frac{K_{IJ}}{4\pi} \partial_x \varphi_I \partial_t \varphi_J - \frac{V_{IJ}}{4\pi} \partial_x \varphi_I \partial_x \varphi_J, \; \varphi_I \sim \varphi_I + 2\pi,$$

with K_{II} even. The canonical conjugate of φ_I is $\frac{K_{IJ}}{2\pi}\partial_x \varphi_J \to$

$$[\varphi_I(x), \varphi_J(y)] = \mathrm{i}\pi (K^{-1})_{IJ} \mathrm{sgn}(x-y)$$

- All the allowed operators have the form $\mathrm{e}^{\mathrm{i}\, l_I \varphi_I(x)}$ where $l_I \in \mathbb{Z}$. The commutation of allowed operators

$$\mathrm{e}^{\mathrm{i}\, l_I \varphi_I(x)} \mathrm{e}^{\mathrm{i}\, \tilde{l}_J \varphi_J(y)} = \mathrm{e}^{\mathrm{i}\pi \tilde{l} K^{-1} l} \mathrm{e}^{\mathrm{i}\, \tilde{l}_J \varphi_J(y)} \mathrm{e}^{\mathrm{i}\, l_I \varphi_I(x)}$$

\- Moving operator $\mathrm{e}^{\mathrm{i}\, l_I \varphi_I(x)}$ around $\mathrm{e}^{\mathrm{i}\, \tilde{l}_J \varphi_J(y)}$ induce a phase $\mathrm{e}^{\mathrm{i}\, 2\pi \tilde{l} K^{-1} l} \to$ **mutual statistics**. The imaginary-time correlation between $\mathrm{e}^{\mathrm{i}\, l_I \varphi_I(x)}$ and $\mathrm{e}^{\mathrm{i}\, \tilde{l}_J \varphi_J(y)}$ has a form

$$\langle \cdots \mathrm{e}^{\mathrm{i}\, l_I \varphi_I(z_1)} \mathrm{e}^{\mathrm{i}\, \tilde{l}_J \varphi_J(z_2)} \cdots \rangle \sim \frac{1}{(z_1 - z_2)^{\gamma} (\bar{z}_1 - \bar{z}_2)^{\bar{\gamma}}}, \quad \gamma - \bar{\gamma} = \tilde{l} K^{-1} l.$$

Fractionalization in general 1d chiral boson theory

Most of the allowed operators $\mathrm{e}^{\mathrm{i}\, l_I \varphi_I(x)}$ are not local (*ie* far away operators do not commute)

- **Local operators**: the operators $\mathrm{e}^{\mathrm{i}\, l_I^{loc} \varphi_I(x)}$ that commute with all allowed operator that are far way:

$$l^{loc} K^{-1} l = \text{even int.} \quad \forall l \in \mathbb{Z} \qquad \to \qquad l_I^{loc} = K_{IJ} n_J.$$

$\mathrm{e}^{\mathrm{i}\, l_I^{loc} \varphi_I(x)}$ corresponds to lattice boson operators.

- The allowed non-local operator $\mathrm{e}^{\mathrm{i}\, l_I \varphi_I(x)}$ create quasi particle with fractional statistics given by $\mathrm{e}^{\mathrm{i}\pi l K^{-1} l}$.
- In fact, the chiral boson model for most K is anomalous, *ie* can not be realized by 1d lattice boson model. But it can be realized by the boundary of 2d FQH Hall state. So the chiral boson model is a edge theory of 2d 2d FQH Hall state.