Lecture #3: Solving Linear Equations

usually, if you have a linear system you want to solve can just use existing software

still, it is important to understand how you could do it by hand

<u>Today</u>: How to make a linear system simpler while preserving its solution(s)

Gaussian Elimination: Adding / subtracting rows from each other

Let's do an example, given the linear system:

$$\begin{aligned} x - y + 2z &= 1 \\ -2x + 2y - 3z &= -1 \\ -3x - y + 2z &= -3 \end{aligned}$$

First put it in matrix-vector form

$$\underbrace{\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -3 \\ -3 & -1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}}_{b}$$

__def:__ the augmented matrix $[A\ b]$

i.e. what we get from appending b to the end

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ \boxed{-2} & 2 & -3 & -1 \\ \boxed{-3} & -1 & 2 & -3 \end{array}\right]$$

helpful to keep track of what the r.h.s. is

__def:__ The pivot $\boxed{}$ in a row is leftmost nonzero

__Goal:__ Make the pivots go from left to right, strictly

__Step #1__: Move the pivot of 2^{nd} & 3^{rd} rows to the right by adding/subtracting 1^{st} row e.g.

$$\begin{array}{lcl} r_2 & = & [-2 \quad 2 \quad -3 \quad -1] \\ +\ 2r_1 & = & 2 * [1 \quad -1 \quad 2 \quad 1] \\ \hline r_2' & = & [0 \quad 0 \quad 1 \quad 1] \end{array}$$

$= r_2 + 2r_1 \Rightarrow r_2 = r_2' - 2r_1$

Poll:

Q: what multiple of r_1 should we add to r_3 to move its pivot to the right?

(a) 2 (b) -2 $\boxed{\text{(c) } 3}$ (d) -1

The new (augmented) matrix is

$$\begin{array}{r} \\ r_2' \to \\ r_3' \to \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & -4 & 8 & 0 \end{array}\right]$$

Now you might be wondering: why are we allowed to do these operations?

Because they preserve the set of all solutions

Let's think about this in our example

First, what is the new linear system?

$$\begin{aligned} x - y + 2z &= 1 \\ z &= 1 \\ -4y + 8z &= 0 \end{aligned}$$

Note: the augmented matrix is just bookkeeping

Lets think about just one operation

Claim: x, y, z satisfy:

$$\begin{array}{l} r_1 \\ r_2 \end{array} \left\{ \begin{aligned} x - y + 2z &= 1 \\ -2x + 2y - 3z &= -1 \end{aligned} \right\} \quad (1)$$

iff they satisfy

$$\begin{array}{l} r_1 \\ r_2' \end{array} \left\{ \begin{aligned} x - y + 2z &= 1 \\ z &= 1 \end{aligned} \right\} \quad (2)$$

Let's actually prove this (always good to try to convince yourself of key facts)

<u>Proof:</u> $\underset{x,y,z}{(1)} \Rightarrow \underset{x,y,z}{(2)}$: This is true b/c we're just taking two equations (r_1 & r_2) we satisfy and adding them to get a new equation that we <u>must</u> also satisfy (r_2') $[r_1 \wedge r_2 \Rightarrow r_1 \wedge r_2 \wedge (r_2')]$ $\Rightarrow r_1 \wedge r_2'$

The more interesting direction is

(2) $\Rightarrow$ (1): i.e. we are not creating new solutions when we replace r_2 with r_2'

The **key** is: the steps are **invertible**

If we have a solution that satisfies r_1 and r_2', we can derive r_2 from r_1 and r_2', so it's also satisfied. $\blacksquare$

The new linear system definitely looks simpler. But when should we stop?

<u>def:</u> An augment matrix is in row echelon (ref) form if all pivots are non-zero and go from left to right

<u>Q2:</u> Is the new augmented matrix in ref?

No, and we'll need a new operation to fix it

We can swap rows: (swap the 2nd and 3rd rows)

$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & \boxed{-4} & 8 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

upper triangular

<u>Q3:</u> why are we allowed this operation?

Again, swap on the rows exactly preserves the set of all solutions

Now that we're in ref finding a solution is easy by back substitution

$$z = 1 \quad (\text{from } 3^{rd} \text{ equation})$$

$$-4y + 8(1) = 0 \Rightarrow y = 2 \quad (\text{from } 2^{nd} \text{ equation})$$

$$x - (2) + 2(1) = 1 \Rightarrow x = 1 \quad (\text{from } 1^{st} \text{ eqn})$$

Now let's connect this back to earlier lectures. In particular, I claimed

$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

has a solution for any b_1 & b_2

Q4: How does Gaussian elimination reveal this fact?

Let's put it in ref:

$$
\begin{array}{rcl}
r_2 & = & [1 \quad 1 \quad b_2] \\
-\frac{1}{2} r_1 & = & -\frac{1}{2}[2 \quad -3 \quad b_1] \\
\hline
r_2' & = & [0 \quad \frac{5}{2} \quad b_2 - \frac{b_1}{2}]
\end{array}
$$

So our equivalent linear system is:

$$
\begin{bmatrix} \boxed{2} & -3 \\ 0 & \boxed{\frac{5}{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - \frac{b_1}{2} \end{bmatrix}
$$

For any b_1 and b_2 you give me, I can backsolve to find x and y

Q5: Is the solution always unique?

Yes, because the ops preserve the set of solutions, and backsubstitution finds here is unique

This is an example where understanding how to do things by hand can help - gives you insights even if you're interested in a family of linear systems

Some linear systems do not have a unique solution

e.g.
$$\begin{bmatrix} 2 & -3 & 0 \\ 1 & 1 & 2 \\ 3 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

what would its ref look like?

$$\begin{bmatrix} \boxed{2} & -3 & 0 \\ 0 & \boxed{\frac{5}{2}} & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

How does back substitution tell you all the solutions?

$0 = 0$ any choice of z is fine!

$$\frac{5}{2} y + 2z = \frac{1}{2}$$

$$2x - 3y = 1$$

Aha! The set of solutions looks like a line in 3-d def: A matrix where you get a row of all zeros in the matrix is called singular

we saw above that square systems with a singular matrix A can have ∞ many solutions

But they can also have no solutions

$$\begin{bmatrix} 2 & -3 & 0 \\ 1 & 1 & 2 \\ 3 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Again, we are revisiting an example from before with new tools

what does its ref look like?

$$\begin{bmatrix} 2 & -3 & 0 \\ 0 & \frac{5}{2} & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \boxed{1} \end{bmatrix}$$

Q6: How does this tell us there is no solution?

Again, via back substitution

$$0 = 1 \quad (3^{rd} \text{ equation})$$

All the mileage we got out of rref is just the begining:

Much easier to get geometric insights about a linear system by first putting it in a convenient normal form

On a related note:

Gauss-Jordan Elimination: Do even more work to make the linear system even simpler

$$\text{ref} \rightsquigarrow \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Every pivot is 1 and above it are only 0s (rref)

Q7: What does backsubstitution do?

$$z = b_3, \quad y = b_2, \quad x = b_1$$

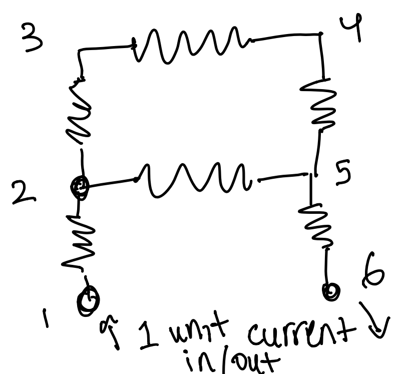
You can't always get the identity matrix, sometimes have to settle for things like

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & \underline{\underline{2}} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

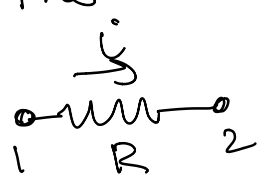
Again, we just use row ops (add/subtract/swap)

In the remaining time let's do an example particularly of how to model problems you might need to solve as a linear system

Ex: Electrical Circuit



we know the current accross a resistor is



$$\frac{V_1 - V_2}{R} = i$$

so we can write down a linear system that describes voltages ⇒ currents

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ & & \vdots & & & \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_6 \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_6 \end{bmatrix}$$

where i_1 = net current out of junction #1

But we know what the current is supposed to be:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

← at all junctures except 1&6, net current should be zero

Hence solving linear system ⇒ fully describing the behavior of the circuit

End note: (for your pset) If A is an $m \times n$ matrix, its transpose denoted A^T is an $n \times m$ matrix with

$$(A^T)_{i,j} = A_{j,i}$$

e.g. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

useful way to reshape a matrix