Lecture 26

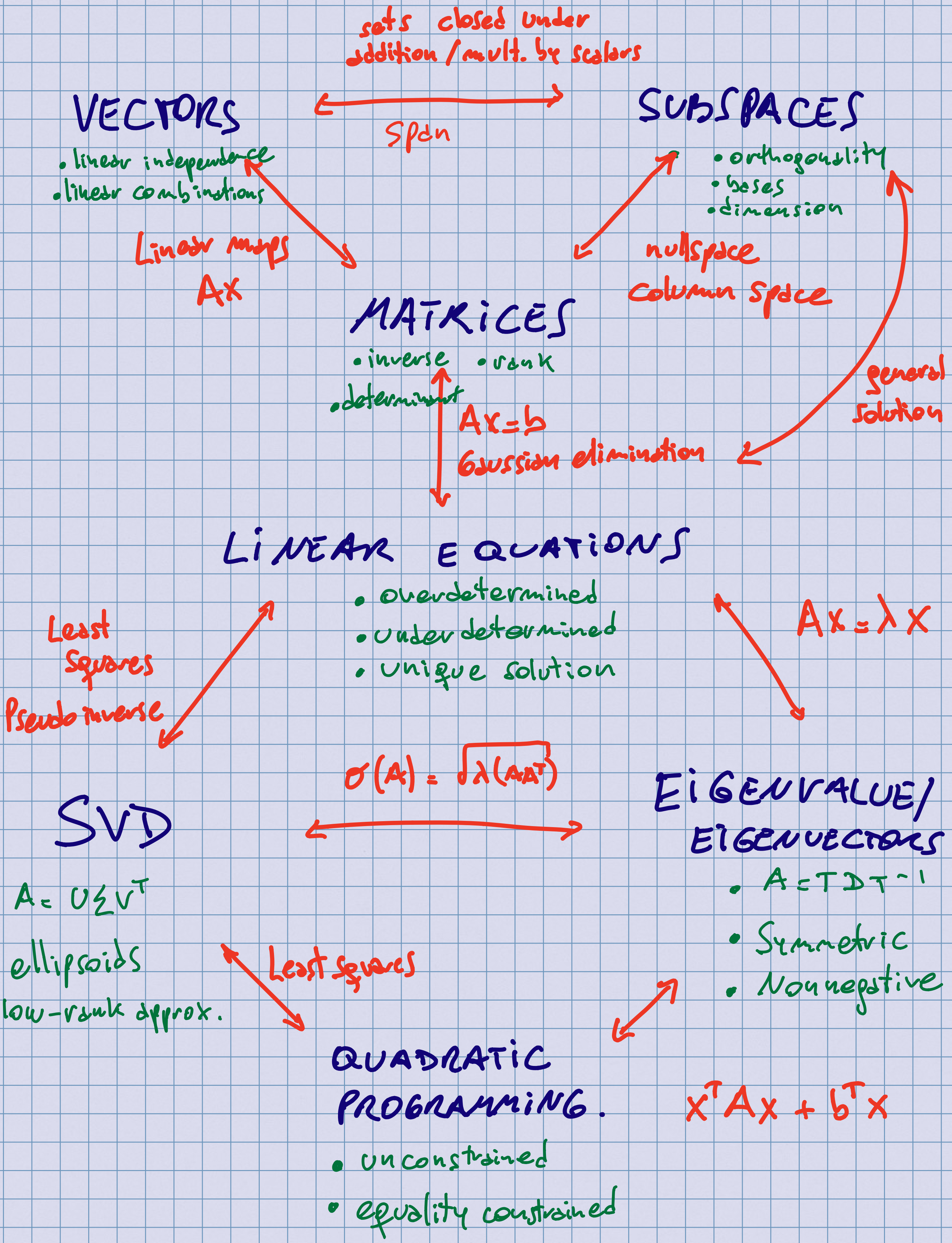
REVIEW

(a bit of everything*)

*DISCLAIMER: NOT COMPLETE!

- PSET 4 is out! Due FRIDAY 9PM.

- MIDTERM FRIDAY 11/13



sets closed under addition / mult. by scalars

Span

VECTORS

- linear independence
- linear combinations

Linear maps

Ax

SUBSPACES

- orthogonality
- bases
- dimension

nullspace

column space

MATRICES

- inverse
- rank
- determinant

$Ax=b$

Gaussian elimination

general solution

LINEAR EQUATIONS

- overdetermined
- underdetermined
- unique solution

Least squares

Pseudo inverse

$Ax = \lambda x$

$\sigma(A) = \sqrt{\lambda(AA^T)}$

SVD

- $A = U\Sigma V^T$
- ellipsoids
- low-rank approx.

EIGENVALUE/ EIGENVECTORS

- $A = TDT^{-1}$
- Symmetric
- Nonnegative

Least squares

QUADRATIC PROGRAMMING.

- unconstrained
- equality constrained

$x^TAx + b^Tx$

VECTORS / MATRICES

x, y, z in vector space V (typical example is $\mathbb{R}^n$)

Addition $x+y \in V$

Scalar multiplication $\lambda x \in V$, $\lambda \in \mathbb{R}$.

Subspaces (lines, planes, hyperplanes ...)

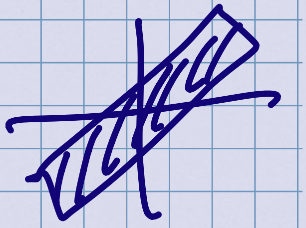
- **LINEAR INDEPENDENCE :** set $\{v_1, \dots, v_k\}$

$$\sum_{i=1}^{k} \lambda_i v_i = 0 \implies \lambda_i = 0$$

- **SPAN :** (all linear combinations)

$$\text{Span}\,\{v_1, .. v_k\} = \{ v \mid \underset{\lambda_i \in \mathbb{R}}{v = \lambda_1 v_1 + \cdots + \lambda_k v_k} \}$$

TWO DESCRIPTIONS OF SUBSPACES.



- **NULLSPACE OF A MATRIX**

  (equations)

$$\underset{\substack{\uparrow \\ \text{subspace}}}{S} = N\big(\underset{\substack{\uparrow \\ \text{matrix}}}{A}\big)$$

$S \subset \mathbb{R}^n$

$A \in \mathbb{R}^{m \times \underline{n}}$

- **COLUMN SPACE OF A MATRIX**

  (generators)

$$\underset{\substack{\uparrow \\ \text{subspace}}}{S} = C\big(\underset{\substack{\uparrow \\ \text{matrix}}}{B}\big)$$

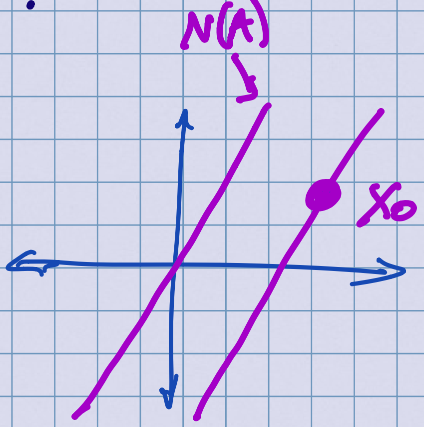
$S \subset \mathbb{R}^n$

$B \in \mathbb{R}^{\underline{n} \times p}$

Can obtain these, and convert to each other, using

RREF / Gaussian elimination.

LINEAR EQUATIONS :



$$\boxed{Ax=b}$$

- General solution: $\underset{\substack{\text{particular}\\ \text{solution}\\ (Ax_0=b)}}{x_0} + \underset{\substack{\text{nullspace}\\ \text{of } A}}{N(A)}$

- Obtain both from RREF/Gauss. Elim.

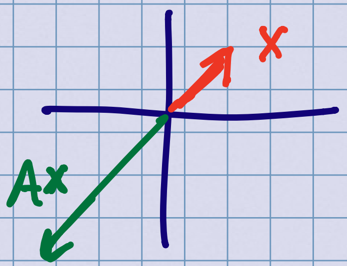
Several possibilities:

- No solution (overdetermined) (e.g. $b \notin C(A)$)
- Unique solution (e.g. A is square and nonsingular)
- Infinite number of solutions (underdetermined) (e.g. $A \in \mathbb{R}^{m\times n}$, $m<n$, A is full row rank)

Eigenvalues / Eigenvectors

$A \in \mathbb{R}^{n\times n}$
(square)

$$\underset{\text{matrix}}{\overset{n\times n}{A}}\;\overset{n\times 1}{x} = \underset{\substack{\text{scalar}\\ \text{(eigenvalue)}}}{\overset{1\times 1}{\lambda}}\;\overset{n\times 1}{x}$$



DIAGONALIZATION:
(sometimes)

$$A = \underset{\text{eigenvectors}}{T}\,\overset{\text{eigenvalues}}{D}\,T^{-1}$$

Singular Value Decomposition (SVD)

$A \in \mathbb{R}^{m\times n}$
(rectangular)

(always exists)

$$\underset{m\times n}{A} = \underset{\substack{m\times m\\ \text{orthogonal}}}{U}\;\underset{\substack{m\times n\\ \text{diagonal}}}{\overset{\substack{\text{singular}\\ \text{values}}}{\Sigma}}\;\underset{\substack{n\times n\\ \text{orthogonal}}}{V^T}$$

(low-rank approximation, PCA, ..)

rank

$$A = \sum_{i=1}^{r} \sigma_i \overbrace{u_i v_i^T}^{\text{rank } 1}$$

(sum of rank 1 matrices)

$$\overbrace{\sigma_1 \geq \cdots \geq \sigma_r}^{\text{nonzero}} > \sigma_{r+1} = \cdots \sigma \cdots = 0$$

USING SVD FOR LINEAR EQS.

$$Ax = b$$

$A \in \mathbb{R}^{m \times n}$

- unique sol? overdetermined? underdet?
- Use SVD: $A = U \Sigma V^T$

$UU^T = I$
$VV^T = I$

$$U \Sigma V^T x = b$$

Define $\tilde{x} = V^T x$
$\tilde{b} = U^T b$

(nonsingular transf.)

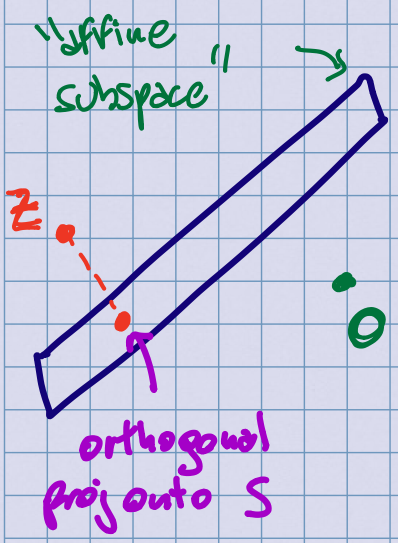
$\Rightarrow$ Much easier! (but same dims)

$$\Sigma \tilde{x} = \tilde{b}$$

$\Sigma \in \mathbb{R}^{m \times n}$

$$\underset{\substack{\uparrow \\ r\ \text{rank}}}{\begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}} \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{bmatrix} = \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \end{bmatrix}$$

USING SVD FOR NULLSPACES | COL SPACES



$$S = \{x \mid Ax = b\}$$

$\left(A \in \mathbb{R}^{m \times n},\ m < n,\ \text{full rank}\right)$

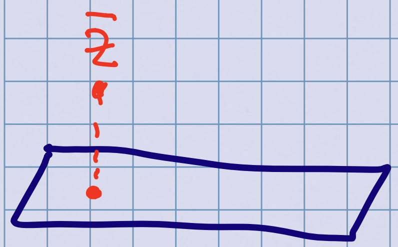
is this a subspace?

How to compute the orthogonal projection of z onto S?

Many good ways.... but let's use SVD!

$$Ax = b \quad \Rightarrow \quad U\Sigma V^T x = b$$

$$\Sigma \underbrace{(V^T x)}_{\tilde{x}} = \underbrace{U^T b}_{\tilde{b}}$$



same thing as projecting $\tilde{z} = V^T x$

onto $\Sigma \tilde{x} = \tilde{b}$

$$\begin{bmatrix}\sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_m & \end{bmatrix}\begin{bmatrix}\bar{x}_1 \\ \vdots \\ \vdots \\ \bar{x}_n\end{bmatrix} = \begin{bmatrix}\bar{b}_1 \\ \vdots \\ \bar{b}_m\end{bmatrix} \qquad \begin{cases}\bar{x}_1 = \bar{b}_1/\sigma_1 \\ \quad\vdots \\ \bar{x}_m = \bar{b}_m/\sigma_m\end{cases}$$

easy to project here!

$$S = \left\{ (a,b,c) \in \mathbb{R}^3 : \begin{array}{l} a = 1 \\ b = 3 \end{array} \right\}$$

orthogonally project $z = (7, 5, -3)$ onto S.

Q: what is the proj of $(7,5,-3)$?

Q: what is the proj of $z = (a,b,c)$?

Symmetric Matrices $A = A^T$

- All eigenvalues are real
- Full basis of orthogonal eigenvectors

$$\Rightarrow \quad A = \overset{\text{eigenvectors}}{\overset{\downarrow}{T}}\, \underset{\underset{\text{eigenvalues}}{}}{\underset{\uparrow}{D}}\, T^T$$

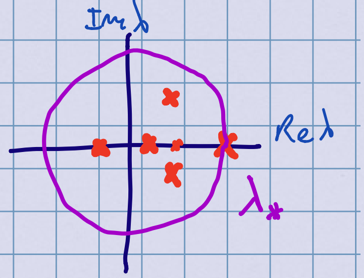
(always diagonalizable)

T is orthogonal $(TT^T = T^T T = I)$

Nonnegative matrices $A_{ij} > 0$

(positive)

- There exists one positive eigenvalue λ^* with positive eigenvector
- λ^* is the largest in magnitude

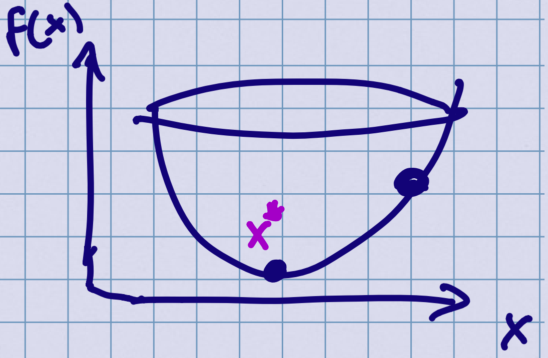


$$|\lambda_i| < \lambda^*$$

and has multiplicity 1.

Related to power method

QUADRATIC PROGRAMMING



- UNCONSTRAINED.

$$\min_{x} \frac{1}{2} x^T \underset{\overset{\uparrow}{\text{symmetric matrix}}}{A} x - b^T x$$

The matrix A is symmetric $\Longrightarrow$ real eigenvalues

Positive (semi) definite if all eigenvalues are positive (nonnegative).

- A is pos def ($\forall \lambda_i \ \lambda_i > 0$) $\Longrightarrow$ opt solution exists and unique
- A is psd and singular $\begin{pmatrix} \forall \lambda_i \ \lambda_i \geq 0 \\ \exists \lambda_i \ \lambda_i = 0 \end{pmatrix}$ $\Longrightarrow$ it depends (eg., linear term)
- A has at least one negative eig ($\exists \lambda_i \ \lambda_i < 0$) $\Longrightarrow$ unbounded below → $-\infty$

OPTIMALITY CONDITION (unconstrained)

$$f(x) = \frac{1}{2} x^T A x - b^T x$$

$$\boxed{\nabla f(x) = 0}$$

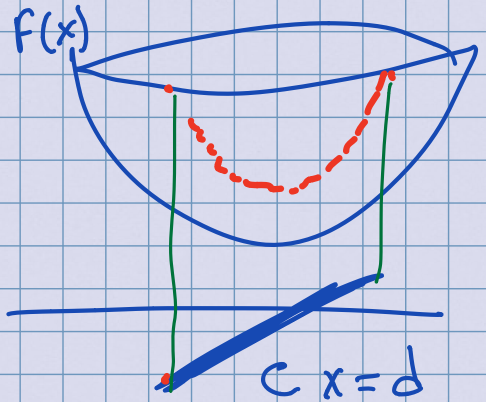
$$\nabla f(x) = Ax - b = 0 \implies \boxed{Ax = b}$$

$$\implies \boxed{x^* = A^{-1} b}$$

$$f(x^*) = \frac{1}{2} (b^T A^{-1}) A (A^{-1} b) - b^T (A^{-1} b)$$

$$= \underline{-\frac{1}{2} b^T A^{-1} b}$$

EQUALITY CONSTRAINED



$$\min \quad \frac{1}{2} x^T A x - b^T x$$

st $\quad Cx = d$

OPT. CONDITIONS

$$\begin{bmatrix} A & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

$Ax + C^T p = b$

$Cx = d$

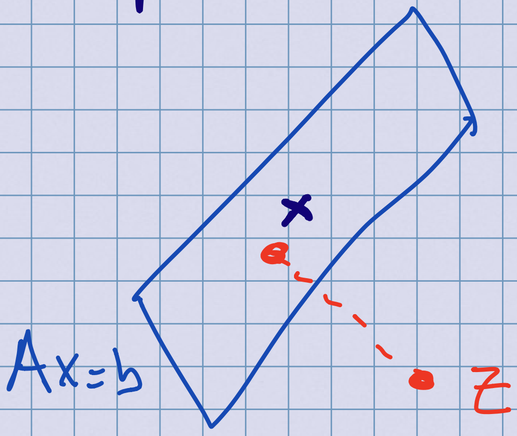
↑ symmetric matrix

↑ auxiliary variable "Lagrange" multiplier

(idea: if x^* is optimal, look at $x^* + z$,
To remain feasible, we must have $z \in N(C)$.)

Exercise :

work out the formula for projection, from the constrained optimization viewpoint



$$\boxed{\begin{aligned} &\min \quad \|x-z\|^2 \\ &\text{s.t.} \quad Ax=b \end{aligned}}$$