6.S084 / 18.S096

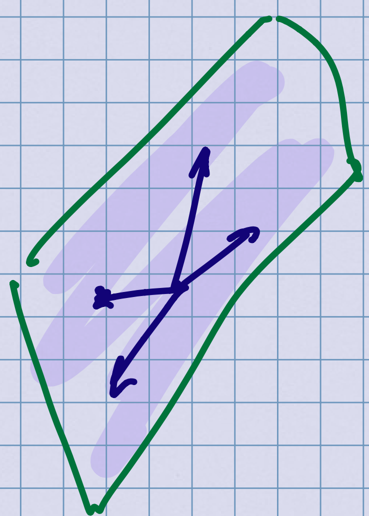
LINEAR ALGEBRA AND OPTIMIZATION

TODAY: Monday 9/21

- Pset Due TODAY 9PM
- Midterm Wed Oct 7
- Check-ups every lecture
- Piazza, Office Hours

Last week Challenge: Find the dimension of the subspace spanned by

$$\left\{ \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \right\}$$



4 vectors in $\mathbb{R}^3$

They span a $\boxed{2\text{-dim}}$ subspace!

Can compute, e.g., using Gaussian elim.

Easy proof: All vectors contained in plane $\boxed{2x - y - z = 0}$.

Lecture 8

The RANK of a matrix

Last time:

- Linear independence $\left(\begin{array}{c} \sum \lambda_i v_i = 0 \\ \Downarrow \\ \lambda_i = 0 \end{array} \right)$

  $+$

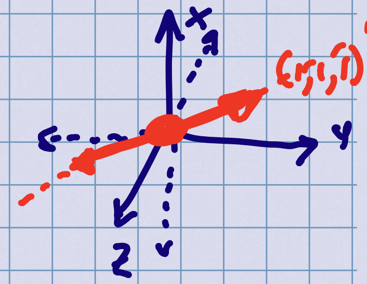
- Generators $\quad \text{span}(v_1, \dots, v_k)$

  $\parallel$

- Bases $\quad \{v_1, \dots, v_k\}$

- Dimension

Subspaces:



Two usual descriptions:

- Generators $S = \text{span}\{v_1, \dots, v_k\}$

$\rightarrow S = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \rightarrow S = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}4\\4\\4\end{pmatrix}\right\}$

- Equations $S = \{x \mid a_i \cdot x = 0\}$

$\{(x,y,z) \in \mathbb{R}^3 : x = y,\ y = z,\ x = z\}$

BASES are very useful.

- But, where do they come from?
- How to compute them?

Claim: If we have generators, can easily "refine" them to a basis.

TASK:

Given subspace $S = \text{span}\{v_1, \dots\dots, v_n\}$
Compute basis $\mathcal{B}$ of S.

Algorithm (basic idea)

Initialize basis : $\mathcal{B} = \{\}$

for k from 1 to n $\qquad \{v_1, \dots v_n\}$

  if v_k is NOT in $\text{span}(\mathcal{B})$

    $\mathcal{B} = \mathcal{B} \cup \{v_k\}$ $\qquad$ (add v_k to basis)

end for

return $\mathcal{B}$.

Claim:

At every step, $\mathcal{B}$ is a basis of $\text{span}\{v_1, \dots, v_k\}$.

We already have an implementation!

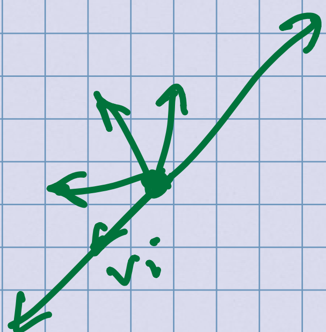
Gaussian elimination

COLUMN RANK : of A

$$m\left\{ A = \begin{bmatrix} \overbrace{v_1 \cdots v_k}^{n} \end{bmatrix} \right.$$

maximum # of LI columns.

$$0 \leq \operatorname{colrank}(A) \leq n$$



ROW RANK :

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$$

maximum # of LI rows

$$\begin{cases} a_1 \cdot X = 0 \\ \vdots \\ a_n \cdot X = 0 \end{cases}$$

$$\begin{cases} x = y \\ x = z \\ y = z \end{cases} \longrightarrow \quad = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$0 \leq \operatorname{rowrank}(A) \leq m$$

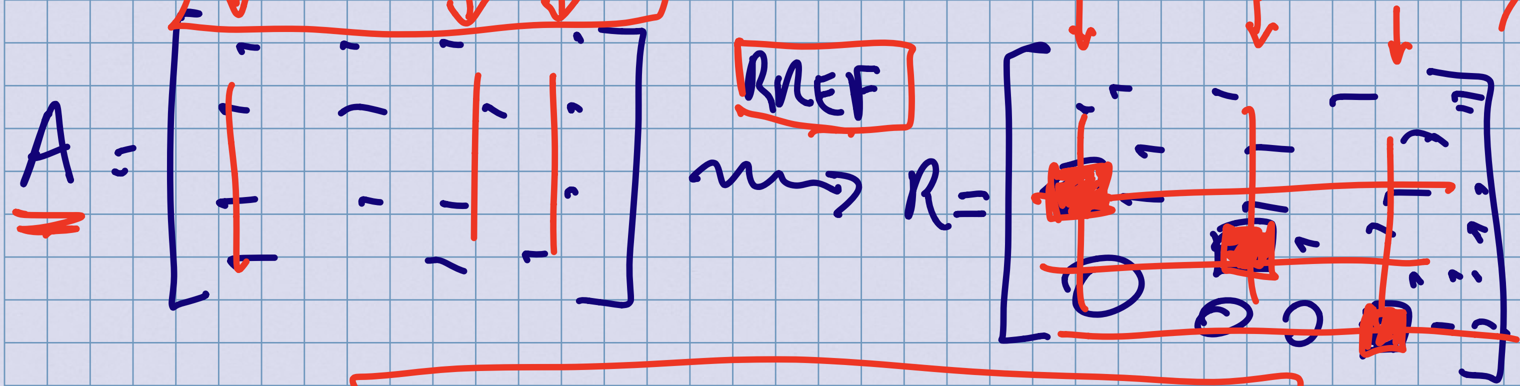
CLAIM:

FOR ANY MATRIX A

$$\boxed{\text{ROW RANK}(A) = \text{COLUMN RANK}(A) = \boxed{\text{RANK}(A)}}$$

Def: The rank of a matrix A is the # of pivots.

$R = EA$



$$\boxed{\text{rank}(A) = \#\,\text{pivots}}$$

and

- pivot columns of A are LI
- pivot columns of R are LI

- pivot rows of R are LI

. A . . . (No!)

Example:

$$E = \begin{pmatrix} 0 & 1/6 & -1/2 \\ 0 & 1/3 & 0 \\ 1 & -1/2 & -1/2 \end{pmatrix}$$

$$A = \begin{bmatrix} -1 & 2 & 2 & 1 \\ 0 & 0 & 3 & 2 \\ -2 & 4 & 1 & 0 \end{bmatrix} \rightsquigarrow R = \begin{bmatrix} \boxed{1} & -2 & 0 & 1/3 \\ 0 & 0 & \boxed{1} & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \text{rank}(A) = 2$$

RANK AND FACTORIZATIONS:

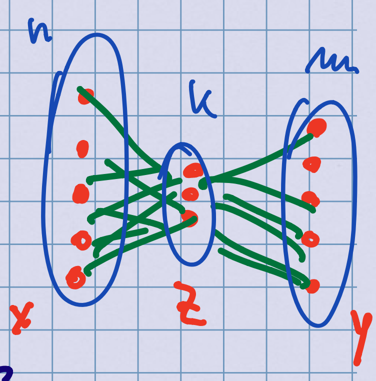
$$\overbrace{A}^{m\times n} = \overbrace{B}^{m\times k} \cdot \overbrace{C}^{k\times n}$$

Rank: minimum k for which such factorization exists

Why do we care?

$$y = Ax = B(\overbrace{Cx}^{z})$$

- Efficient algorithms (e.g. MatMult).
- Dimensionality reduction



- Recomendation systems.
  (e.g., Netflix).

(i) Let A be a $m \times n$ matrix

($m \geq n$, "tall matrix")

$$\begin{array}{cc} & \overbrace{\hphantom{\quad a_{ij} \quad}}^{n \text{ columns}} \\ m \text{ rows} \left\{ \right. & \begin{bmatrix} & & \\ & a_{ij} & \\ & & \end{bmatrix} \end{array}$$

Thm: The following are equivalent:

- The columns of A are linearly independent
- If solvable, $Ax = b$ has a unique solution
- A has a left inverse
- $N(A) = \{0\}$
- $\text{rank}(A) = n$ ("full column rank")

(ii) Let A be a $m \times n$ matrix

($m \leq n$, "fat" matrix)

$$m \text{ rows} \left\{ \overbrace{\begin{bmatrix} \quad & a_{ij} & \quad \end{bmatrix}}^{n \text{ columns}} \right.$$

Thm: The following are equivalent:

- The rows of A are linearly independent
- $Ax = b$ is solvable for every b
- A has a right inverse
- $C(A) = \mathbb{R}^m$
- $\text{rank}(A) = m$ ("full row rank")

(iii)

Let A be a square $n \times n$ matrix

$$\begin{array}{cc} & \\ n \text{ rows} \left\{ \right. & \begin{bmatrix} a_{ij} \end{bmatrix} \\ & \underbrace{\hphantom{[\ a_{ij}\ ]}}_{n \text{ columns}} \end{array}$$

Thm: The following are equivalent:

- A is invertible (nonsingular) A^{-1} exists
- $Ax = b$ always has a unique solution
- $N(A) = \{0\}$
- $C(A) = \mathbb{R}^n$
- $\operatorname{rank}(A) = n$

<u>More generally</u>:

$$\boxed{A \in \mathbb{R}^{m \times n}}$$

"Rank-Nullity" Theorem.

$$\boxed{\underbrace{\text{rank}(A)} + \underbrace{\dim N(A)} = n}$$

(more about this later in the course)

PUZZLE!

Alice "proves":

A has a left inverse $\xRightarrow{\text{(false!)}}$ $Ax=b$ is always solvable

(NOT A)
"Proof": $\exists N: \quad NA = I$ (exist left inv)

$$Ax=b \Rightarrow \overbrace{NA}^{I}x = Nb \Rightarrow \boxed{x=Nb}$$ ∎

Q: What's wrong with this ???