Lecture #2: The Geometry of Linear Equations

Today: Many different ways to think about systems of linear equations

Start with a running example:

$$2x - 3y = 0$$
$$x + y = 5$$

two equations, two unknowns

We can write it in matrix-vector notation:

$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

Last time we talked about matrix-vector products and now we're using a vector of variables

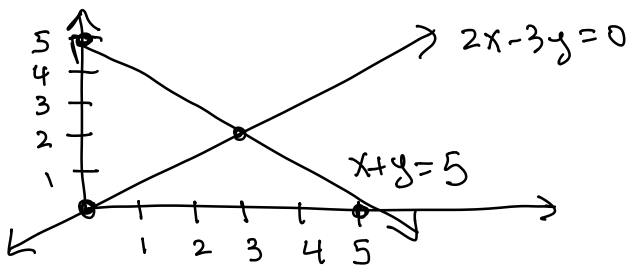
<u>Interpretation #1</u>: The row view

Think of matrix-vector product as taking inner-product of rows of matrix and vector

e.g. $\begin{bmatrix} 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x - 3y = 0$

Each row $\longleftrightarrow$ linear constraint

Let's draw these so we can visualize it



What can we learn from this visual?

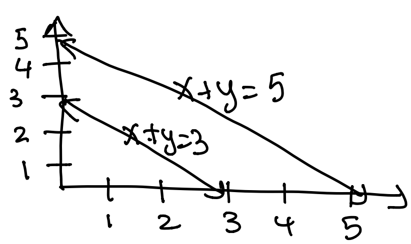
Q1: Can you give me a solution? (3,2)

Q2: Is the solution unique?

This is what we expect to typically happen when

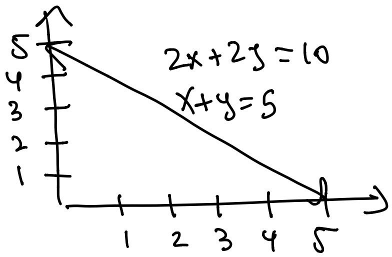
 # variables = # constraints

Q3: What would it look like if there was no solution?



The two equations are contradictory

Q4: What would it look like if there was more than one solution?



$x + y = 5$
$2x + 2y = 10$

The constraints are redundant

poll <u>Q5</u>: Can you ever have the # of solutions be anything other than $0, 1,$ or ∞?
No

Many questions in linear algebra are helpful to think about multiple ways

<u>Interpretation #2</u>: The Column View

Can also view a matrix-vector product as a linear combination of columns of the matrix

e.g. $\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Longleftrightarrow$

<u>def:</u> A <u>linear combination</u> (over the reals) of the vectors $v_1, v_2 \ldots v_m$ is a sum

$$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m$$

where each c_i is a real number

$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 5 \end{bmatrix}}_{b}$$

This gives us a new way to think about linear systems:

"Is there a linear combination of

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

that equals $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$?"

Now let's use the column view to get some new insights

<u>Q6</u>: Is there a linear combination of

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

that equals $\begin{bmatrix} \pi \\ \zeta(3) \end{bmatrix}$?

Riemann zeta function

<u>Theorem</u> [Apéry]: $\zeta(3)$ is irrational

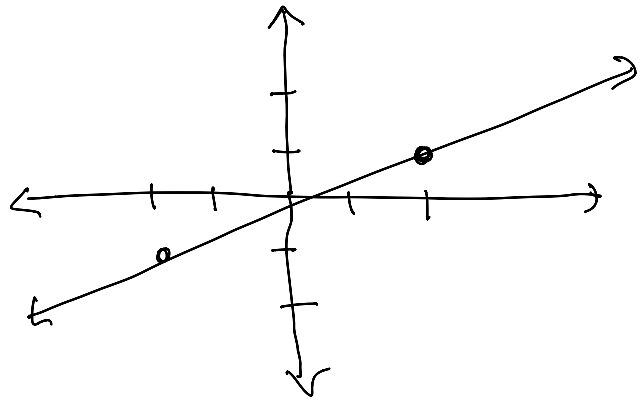
Thankfully, this is not relevant b/c it doesn't matter what the right hand side is!

Again, let's visualize it: what does the <u>set</u> of vectors we can get as linear combinations of

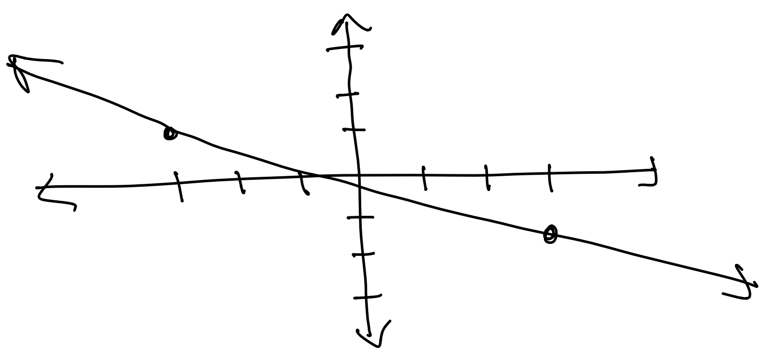
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

look like?

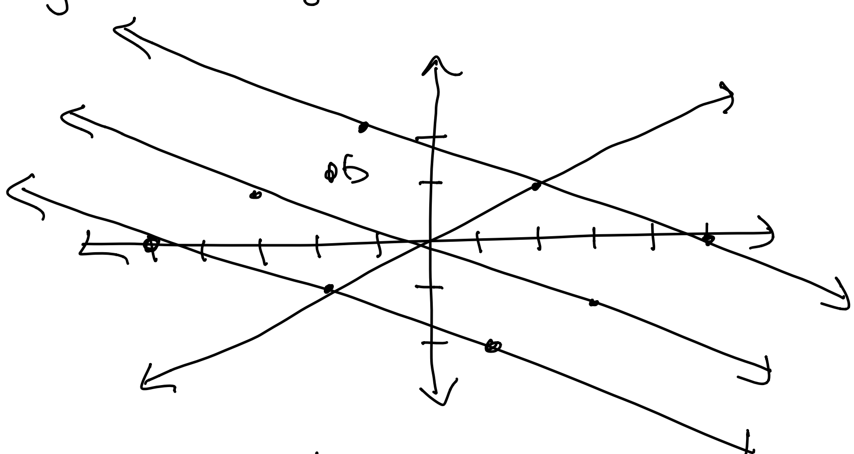
Let's start with all linear combinations of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$



What about all vectors that are linear combinations of just $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$?



Putting it all together we get a skewed grid



and everything in between – all of $\mathbb{R}^2$

It would have been much easier if I asked what you could get as linear combinations of

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

But they are actually the same question!

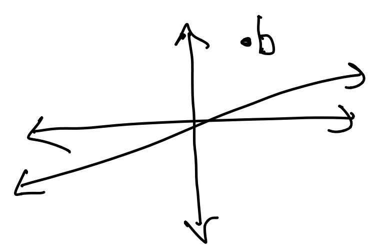
Q7: Are there 2-dimensional vectors v_1 and v_2 s.t. you can't get the entire plane as linear combinations of them?

A helpful definition

def: The span of vectors $v_1, v_2 \ldots v_m$ is all the vectors that can be obtained as linear combinations of $v_1 \ldots v_m$

Q7': Are there 2-dimensional vectors v_1 and v_2 that don't span the plane? Yes

e.g. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$



The second line falls on top, so you don't get a grid

Things are more subtle for 3x3 and larger
(3,3) in Julia

Let's revisit earlier questions

Q3: what does it look like if there are no solutions?

In the 2x2 case, this only happened when two rows were the same after rescaling

e.g.

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$$

$\times \frac{1}{2}$ $\Updownarrow$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

In particular, rescaling a row and the constraint does not change the set of solutions

But more complicated things happen in general

Let's see, through an example:

$$\begin{bmatrix} 2 & -3 & 0 \\ 1 & 1 & 2 \\ 3 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Are there any rows that are scalar multiples of each other?

It's harder to draw in 3-d (and don't even try say 10-d), but I can still convince you there is no solution

poll what equation does the first row correspond to?

$$\begin{array}{rrrrl} & 2x & -3y & & = 1 \\ + & x & +y & +2z & = 1 \\ \hline & 3x & -2y & +2z & = 2 \end{array}$$

but we have the constraint

$$3x - 2y + 2z = 3$$

We had three equations and three unknowns. what went wrong, from the column perspective?

well, the columns of the matrix do not span $\mathbb{R}^3$

Is there a way to see this?

the column $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ is itself a linear combination of $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$

<u>Exercise</u>: Express $\begin{bmatrix}0\\2\\2\end{bmatrix}$ as a linear comb. of $\begin{bmatrix}2\\1\\3\end{bmatrix}$ and $\begin{bmatrix}-3\\1\\-2\end{bmatrix}$

<u>def:</u> A set of vectors $v_1, v_2 \ldots v_m$ is <u>linearly independent</u> if no v_i can be written as a linear combination of the others