Lecture #6: The Column Space & Nullspace

Today we will explore a key definition, and how it helps us think about matrices as operators in new ways

<u>def:</u> A <u>vector space</u> over the reals is a set V with notions of

① <u>how to add</u>: $\forall\, v, w \in V,\ v + w \in V$

② <u>how to scale</u>: $\forall\, v \in V,\ \forall\, \alpha \in \mathbb{R},\ \alpha v \in V$

we have already seen some examples

$$V = \mathbb{R}^d = \text{set of all } d\text{-tuples of reals}$$

In the first lecture we talked about how to add and multiply by a scalar

Often we will get geometric insights about matrices by understanding fundamental vectorspaces associated with them

<u>def:</u> If V is a vector space then $S \subseteq V$ is a <u>subspace</u> if

① $\forall\, v, w \in S,\ v + w \in S$

② $\forall\, v \in S,\ \forall\, \alpha \in \mathbb{R},\ \alpha v \in S$

Q: What are the subspaces of $\mathbb{R}^2$?

Let me give you the easy ones: $S = \{0\}$, $S = \mathbb{R}^2$

Anything else?

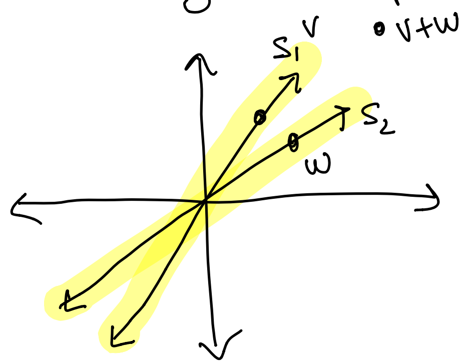
$S =$ any line passing thru the origin

A key property of vector spaces/subspaces is that you can combine them to get new ones

Let's explore what we can/cannot do:

Q2: If S_1 and S_2 are two subspaces of $\mathbb{R}^2$ is $S_1 \cup S_2$ necessarily a subspace?

No



$v+w$

Q3: Is $S_1 \cap S_2$ necessarily a subspace?

Yes!

Fact: If S_1 and S_2 are subspaces, then so is $S_1 \cap S_2 = S$

It is easy to verify the properties of a subspace:

(1) suppose $v, w \in S_1 \cap S_2$ then

$$\left.\begin{array}{l} v, w \in S_1 \Rightarrow v+w \in S_1 \\ v, w \in S_2 \Rightarrow v+w \in S_2 \end{array}\right\} \Rightarrow v+w \in S_1 \cap S_2$$

(2) suppose $v \in S_1 \cap S_2$ and $\alpha \in \mathbb{R}$, then

$$\left.\begin{array}{l} v \in S_1 \Rightarrow \alpha v \in S_1 \\ v \in S_2 \Rightarrow \alpha v \in S_2 \end{array}\right\} \Rightarrow \alpha v \in S_1 \cap S_2$$

Now let's introduce a first fundamental subspace:

def: Given a matrix A with columns $a_1, a_2 \ldots a_n$ then the column space of, denote $C(A)$ is

$$C(A) = \{\alpha_1 a_1 + \alpha_2 a_2 \ldots + \alpha_n a_n \mid \alpha_i \in \mathbb{R}\}$$

To put it another way, it's the set of all linear combinations of columns of A (i.e. their span)

Actually, we've been talking about this implicitly for a while, e.g.

In the second lecture we said

$$\underbrace{\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

has a solution for any b_1 and b_2.

Later we reasoned about this example via Gaussian elimination and inverses

Alternatively $C(A) = \mathbb{R}^2$

More generally:

Fact: $A\underset{\uparrow}{x} = \underset{\uparrow}{b}$ has a solution for some b iff $b \in C(A)$

vectors

Now let's understand the columnspace thru matrix inverses. Recall

Fact: If A is square and invertible then its right and left inverses are the same

i.e $BA = I \Rightarrow AB = I$ and vice versa

Now here's another way to think about invertibility:

Fact: If A is square and $n \times n$ then

$$A \text{ is invertible iff } C(A) = \mathbb{R}^n$$

This is important so let's think about why this is true:

$\Rightarrow$

① Suppose A is invertible. I want to show that for any $b \in \mathbb{R}^n$ it can be expressed as a linear combination of columns in A

So what linear combination should I use?

Set $x = A^{-1}b$. Then $Ax = (AA^{-1})b = I \cdot b = b$

$\Leftarrow$

② Conversely, what if $C(A) = \mathbb{R}^n$? How can I find the inverse of A?

Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, etc standard basis vectors

Now if $e_1 \in C(A)$ there must be x_1 with

$$Ax_1 = e_1, \text{ etc}$$

Now consider $A\underbrace{[x_1\ x_2\ \dots\ x_n]}_{A^{-1}} = [e_1\ e_2\ \dots\ e_n]$

Thus we have found the inverse of A

<u>side note:</u> Now that we have many linear algebraic concepts under our belt it is helpful to go back and understand old tools in a new language, just like we did above

Now let's talk about our second fundamental subspace:

<u>def.</u> The <u>nullspace</u> of A, denote $N(A)$ is

$$N(A) = \{x \mid Ax = 0\}$$

<u>Q4:</u> What if I take some b and look at

$$S = \{x \mid Ax = b\} ?$$

Is S a subspace? No

$$Ax_1 = b, \ Ax_2 = b \Rightarrow A(x_1 + x_2) = 2b$$

In general, it looks like a line, plane, etc but not through the origin

The columnspace and the nullspace are very inter-related

Q5: If A is square and there is a nonzero vector $x \in N(A)$, can A be invertible?

No, b/c invertibility means you can undo multiplying by A, but if $x \neq 0$ and $Ax = 0$ then

$$Ax = 0 = A \cdot \underset{\text{all zero vector}}{0}$$

Again, let's revisit an earlier example with our new definitions

$$A = \begin{bmatrix} 2 & -3 & 0 \\ 1 & 1 & 2 \\ 3 & -2 & 2 \end{bmatrix}$$

Let's put it in reduced row echelon form:

$$A = \underset{\text{what does B represent?}}{B} \underbrace{\begin{bmatrix} 1 & 0 & \frac{6}{5} \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix}}_{A'}$$

Can I use this expression to find a nonzero x with $Ax = 0$?

$$Ax = 0 \iff A'x = 0$$

Well, let's find x with $A'x = 0$. That will work for our original question

$$\underbrace{\begin{bmatrix} 1 & 0 & \frac{6}{5} \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix}}_{A'} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + \frac{6}{5}z \\ y + \frac{4}{5}z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

try $x = 1$, $z = -\frac{5}{6}$, $y = \frac{2}{3}$

To be clear, we're making use of the following strategy:

<u>Fact:</u> If B is square and invertible then

$$N(A') = N(BA')$$

This is b/c $A'x$ is zero iff $BA'x$ is zero

(see Q5)

Thus Gauss-Jordan Elimination also gives us a way to find the nullspace

Just to drive the point home, let's connect this all back to the column space

Q7: why does there being a nonzero vector in the Nullspace mean $C(A) \neq \mathbb{R}^n$?

In our example:

$$\text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\right) = \text{line}$$

$$\text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}\right) = \text{plane}$$

But now consider a general linear combination of all three columns:

$$\alpha_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \qquad (*)$$

How do we argue we still just get a plane?

We know that

$$\frac{5}{6} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$$

from what we computed about $N(A)$

Hence we can rewrite (*) as:

$$\left(\alpha_1 + \frac{6}{5}\alpha_3\right)\begin{bmatrix}2\\1\\3\end{bmatrix} + \left(\alpha_2 + \frac{4}{5}\alpha_3\right)\begin{bmatrix}-3\\1\\-2\end{bmatrix}$$

which doesn't give us anything new — still just a plane

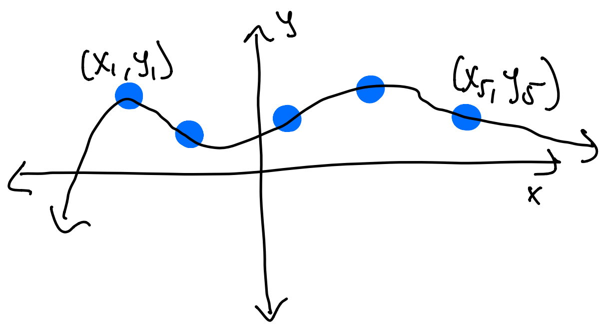
Putting it all together:

<u>Fact:</u> If A is square and $n \times n$ then

$$C(A) = \mathbb{R}^n \iff N(A) = \{0\}$$

<u>Application:</u> (over) fitting with polynomials

Imagine I have a bunch of data in 2-d



Can we find a polynomial $p(x) = c_0 + c_1 x \ldots + c_d x^d$ that fits the data?

we can write this as a linear system:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^d \\ 1 & x_2 & x_2^2 & \cdots & x_2^d \\ \vdots & & & & \vdots \\ 1 & x_5 & x_5^2 & \cdots & x_5^d \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_d \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_5 \end{bmatrix}$$

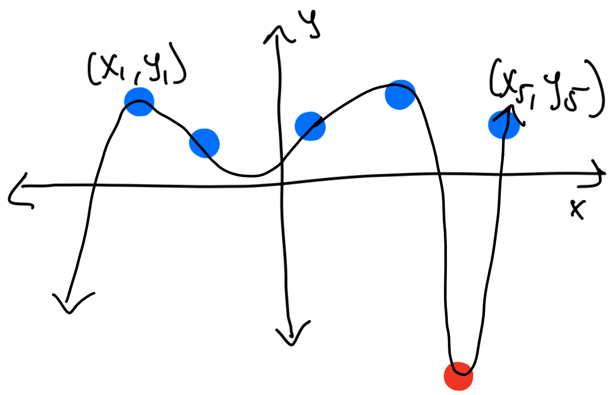
$5 \times (d+1)$

$$A \quad c \quad = y$$

what if I find a polynomial that fits? what should I be worried about in $N(A) \neq \{0\}$?

Take $e \in N(A)$. Not only does the polynomial corresponding to c fit, but also $c+e$, etc

So maybe I can make it do this



Linear algebra plays a key role
in understanding if you are over-
fitting, i.e. via statistical
significance