6.S084 / 18.S096

LINEAR ALGEBRA AND OPTIMIZATION

TODAY: 9/18

- Pset Due Monday 9/21
- Midterm Wed Oct 7
- Check-ups every lecture
- Piazza, Office Hours

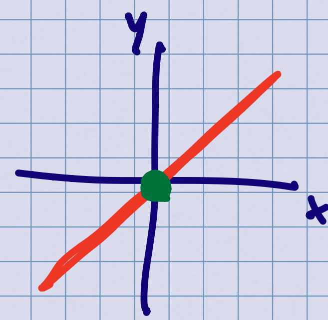
Lecture 7

Linear Independence, Dimension, Bases.

Last time: subspaces.

nullspace $N(A)$

column space $C(A)$

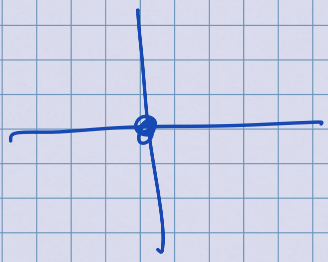
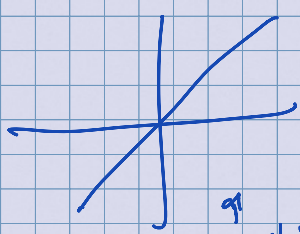
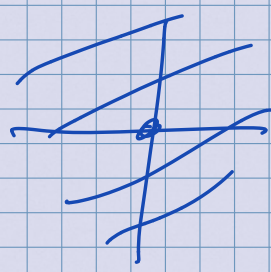


$\{x \mid Ax = 0\} = N(A)$ $\quad A \in \mathbb{R}^{m \times n}$

$A = [v_1 \cdots v_n]$

$C(A) = \sum \lambda_i v_i$

Want to understand their properties (e.g., how "big" they are)

(Lec 2).

Recall the definition of linear independence

"no v_i can be written as a linear combination of other v_j".

Def: Given $\{v_1, v_2, \dots v_k\}$, they are LI if

$$\sum \lambda_i v_i = 0 \implies \lambda_i = 0 \quad \forall i$$

$\lambda_i \in \mathbb{R}$

$\{v_1, v_2, \dots v_k\}$ $\quad \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_k v_k = 0$

$\hookrightarrow$ NOT LI: $\quad \lambda_1 \neq 0 \quad v_1 = -\left(\frac{\lambda_2}{\lambda_1}\right) v_2$

Def: $\{v_1 \dots v_k\}$ is linearly dependent (LD) if they are not LI.

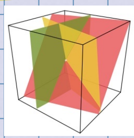
(aside)

Q: What does this mean?

Q: Why do we care?

In this course, we want to understand things:

- Algebraically (Equations) $Ax = b$
- Geometrically (Figures)

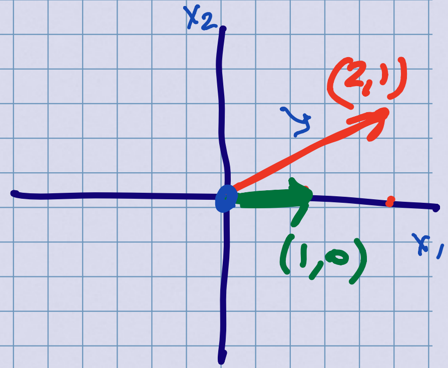


- Computationally (algorithms + code)

julia

LI: $\underbrace{\lambda_1 v_1 + \cdots + \lambda_k v_k} = 0 \Rightarrow \lambda_i = 0$

Ex 1: $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

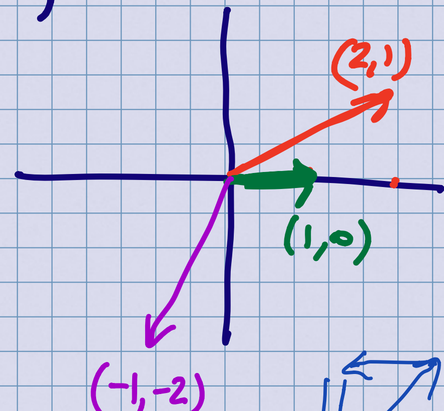


$$\lambda_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2\lambda_1 + \lambda_2 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases} \Rightarrow \text{LI}$$

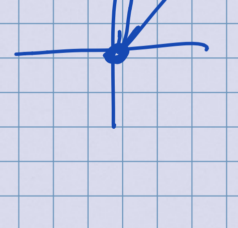
Ex 2: $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \end{pmatrix} \right\}$

$$\lambda_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



$$\boxed{\lambda_1 = 2 \quad \lambda_2 = -3 \quad \lambda_3 = 1}$$

$\hookrightarrow \boxed{\text{LD}}$



Equivalent forms

$$\boxed{\sum \lambda_i v_i = 0}$$

$$\boxed{\underbrace{\begin{pmatrix} v_1 & \cdots & v_k \end{pmatrix}}_{A} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}}$$

$$\underbrace{A\lambda = 0}_{\uparrow \; \boxed{\lambda = 0}}$$

$\{v_1 \dots v_k\}$ are linearly independent.

$$\Updownarrow$$

$A\lambda = 0$ has a unique solution.

$$\Updownarrow$$

Nullspace of A $= \{0\}$.

CAN DO WITH GAUSSIAN ELIMINATION

Recall that:

- if T is invertible, then

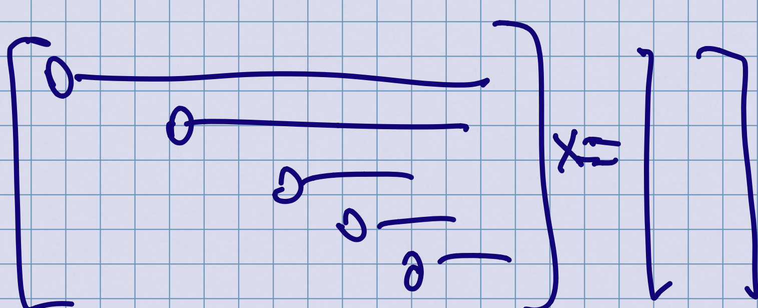
$$\boxed{N(A) = N(TA)}$$

$$x \in N(A) \iff Ax = 0 \iff TAx = T0 = 0$$

$$\Updownarrow$$

$$x \in N(TA)$$

- FINDING THE NULLSPACE OF A MATRIX in RREF FORM IS EASY. (WHY?).



EXAMPLE:

$$\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \end{pmatrix} \right\}$$

$$\boxed{A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & -2 \end{pmatrix}} \xrightarrow{\text{RREF}} \underbrace{\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}}_{TA}$$

$N(A)$

$N(TA)$

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & -2 \end{pmatrix}}_{A} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\Rightarrow N(A) = N(TA) = \left\{ \begin{pmatrix} 2\alpha \\ -3\alpha \\ \alpha \end{pmatrix} : \forall \alpha \in \mathbb{R} \right\}$$

$\Rightarrow$ vectors are LD.

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \begin{cases} x_1 - 2x_3 = 0 \\ x_2 + 3x_3 = 0 \end{cases}$$

Generators. (of a subspace).

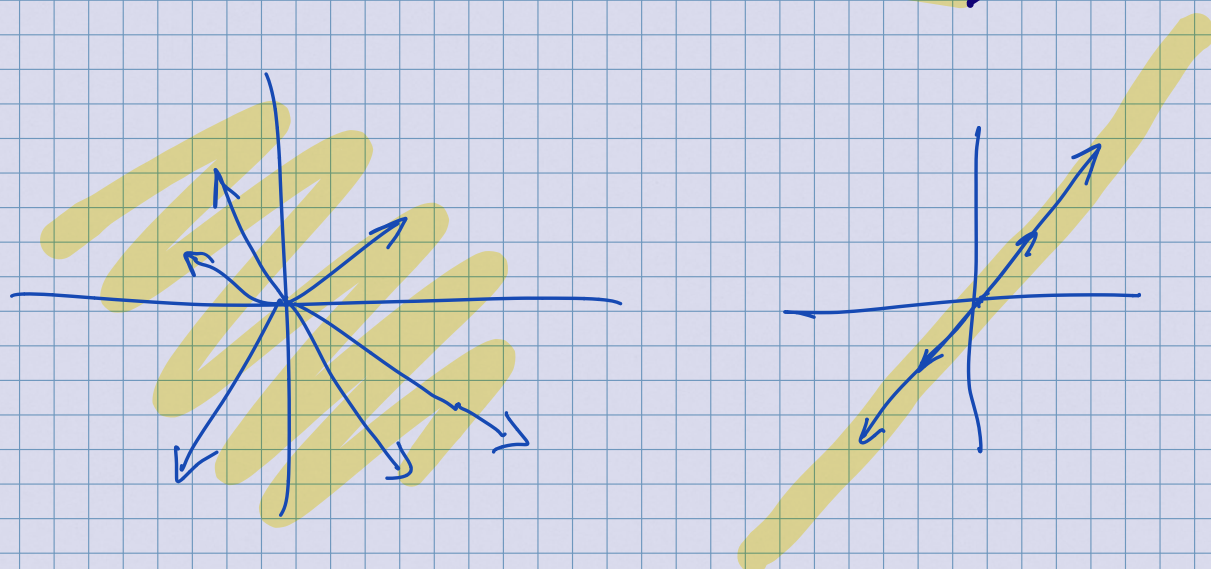
S

Def: $\{v_1, \dots v_k\}$ are generators of S if every vector $v \in S$ can be written as

$$v = \lambda_1 v_1 + \cdots + \lambda_k v_k .$$

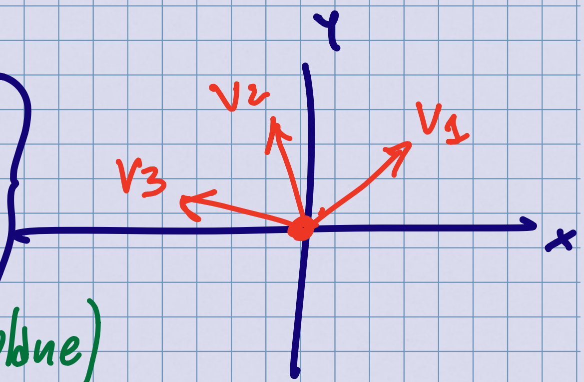
Notation: $S = \langle v_1, \dots, v_k \rangle$

or $S = \mathrm{span}(v_1, \dots, v_k)$



Geometrically:

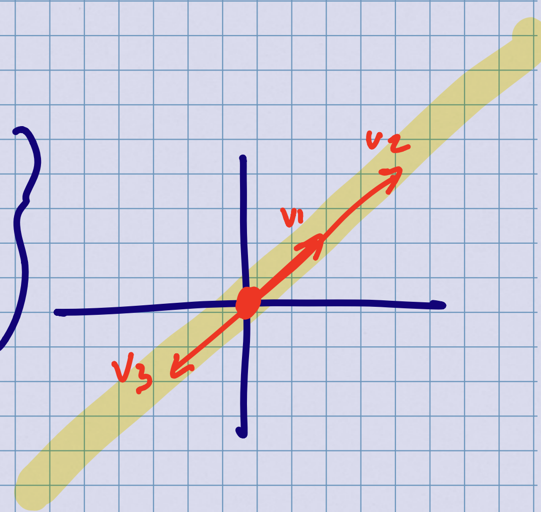
$$\text{span}\left\{\begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}-1\\3\end{pmatrix}, \begin{pmatrix}-3\\1\end{pmatrix}\right\}$$



$= \mathbb{R}^2$ (the whole plane)

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = \lambda_1\begin{pmatrix}2\\2\end{pmatrix} + \lambda_2\begin{pmatrix}-1\\3\end{pmatrix} + \lambda_3\begin{pmatrix}-3\\1\end{pmatrix}$$

$$\text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\right\}$$



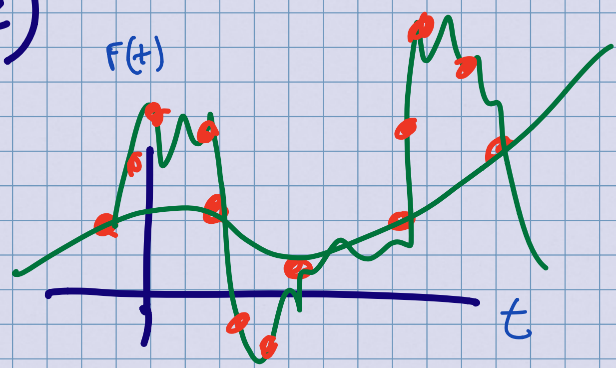
$= \alpha \cdot \begin{pmatrix}1\\1\end{pmatrix}$ (a line)

Why do we care? (e.g., in applications)

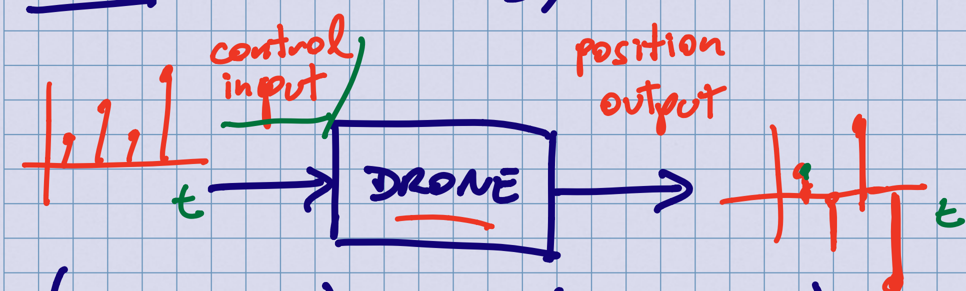
Ex 1: data fitting, interpolation (last time)

$$f(t) = \sum \alpha_i f_i(t)$$

f_i given functions.



Ex 2: robotics, control



$$\underbrace{(u_1, u_2, \ldots u_k)}_{u} \qquad \underbrace{(y_1, y_2, \ldots, y_\ell)}_{y}$$

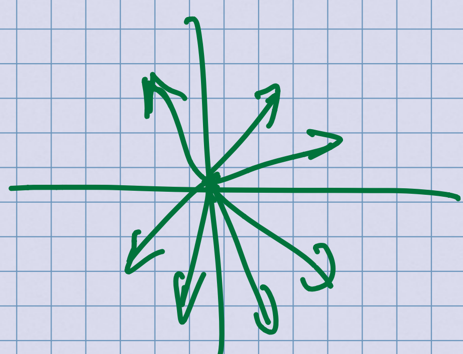
$$\boxed{y = Gu}$$

Q: What are <u>all</u> the possible outputs I can obtain?

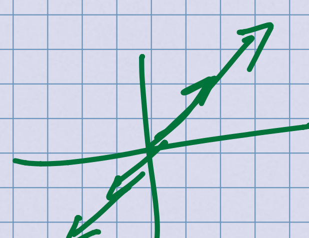
"reachable subspace of a dynamical system".

Q: How "big" is it?

Generators are great, but they can be "wasteful". (e.g., Example)



$\{ v_1, \ldots \ldots \; v_{17} \}$



Def: $\overbrace{\{v_1, \dots v_k\}}$ is a **basis** of subspace S, if they generate S and they are LI.

Ex: "canonical basis" of $\mathbb{R}^n$

$\{e_1, \dots e_n\}$ is a basis of $\mathbb{R}^n$

$$n\left\{\underbrace{\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{n}\right.$$

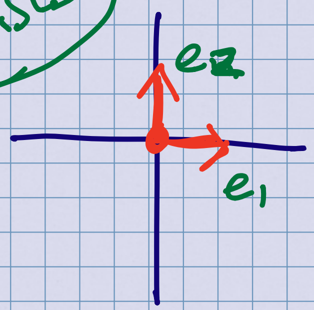
→ generates $\mathbb{R}^n$ ✓

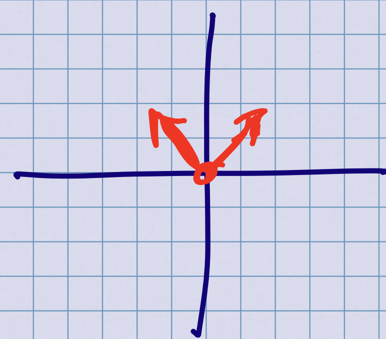
$x = \begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} \Leftrightarrow x = \sum x_i e_i$

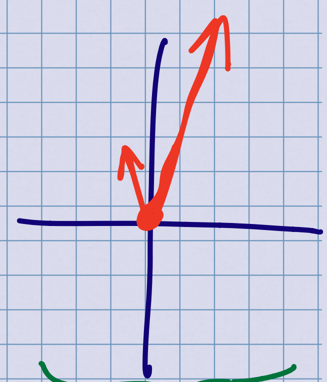
→ LI ✓

Ex:

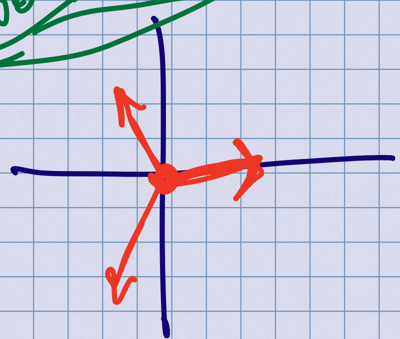
BASES of $\mathbb{R}^2$



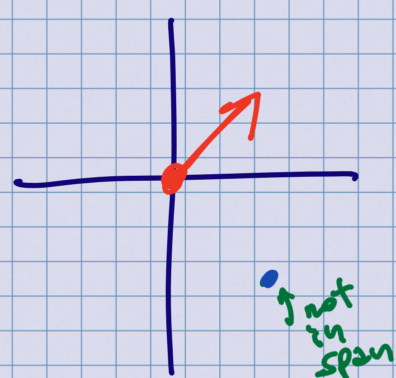




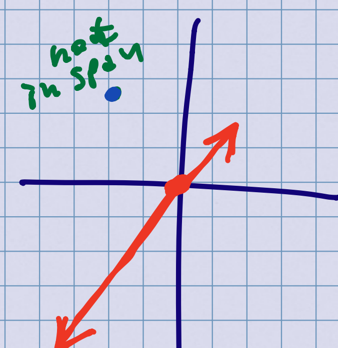
NOT BASES OF $\mathbb{R}^2$



NOT LI



Does not generate $\mathbb{R}^2$



Does not generate $\mathbb{R}^2$.

Nice. But, what if we have one basis with 3 elements, and another with 8.??

CANNOT HAPPEN!

Thm All bases of S have the same cardinality. (# of vectors).

Def: The dimension of a subspace is the cardinality of any basis.

Why do we care?

(e.g., robotics example)

Q: How do we computationally determine the dimension of a subspace?

HINT: Gaussian elimination!

Challenge: Find the dimension of the subspace spanned by

$$\left\{ \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \right\}$$