Lecture 10

The Determinant and its Properties

- PSET 2 OUT!
  Due Oct 5

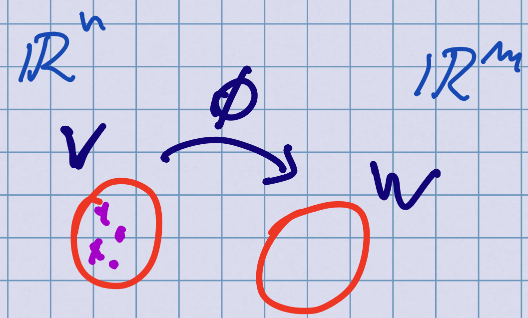
- CHECKUPS
  please do them!. Let us know if something's funky!

Linear transformations

(or linear maps).

Linear functions between vector spaces

$$\phi : V \longrightarrow W$$



Formally: ϕ must satisfy

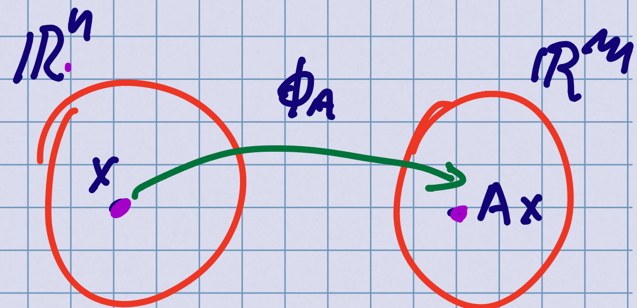
$$\boxed{\phi(\alpha x + \beta y) = \alpha \phi(x) + \beta \phi(y) \quad \forall \alpha, \beta \in \mathbb{R}, \quad x, y \in V}$$

(typechecks)

We've seen this before!

In particular, any <u>matrix</u> $A \in \mathbb{R}^{m \times n}$ <u>defines</u> a linear map.

$$\begin{aligned} \phi_A : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ x &\longmapsto Ax \end{aligned}$$



<u>Ex:</u>

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \mathbb{R}^{2 \times 3}$$

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbb{R}^3} \longmapsto \underbrace{\begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_3 \end{bmatrix}}_{\mathbb{R}^2}$$

Q: So, what's the difference? (if any)

Why do we talk about both

$\boxed{\phi_A}$ linear map

and ?

$\boxed{A}$ matrix

vs.

"rabbit" (english)

"conejo" (spanish)

"lapin" (french)

"Hase" (german)

⋮

exists independently

"intrinsic"

depends on ~~speaker~~ bases.

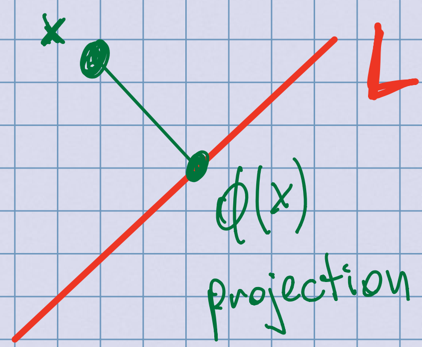
linear map

$$V \xrightarrow{\phi} W$$

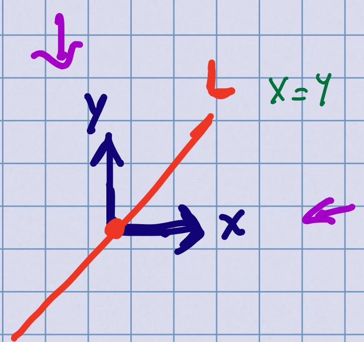
matrix

$$A = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Example: projection onto a line

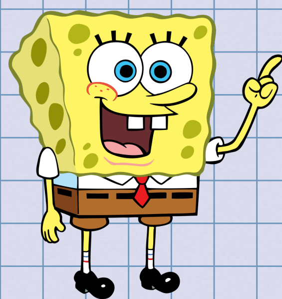


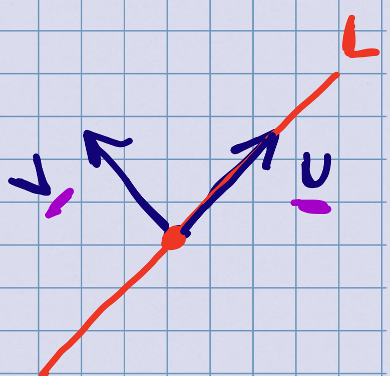
Alice:



$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Bob:



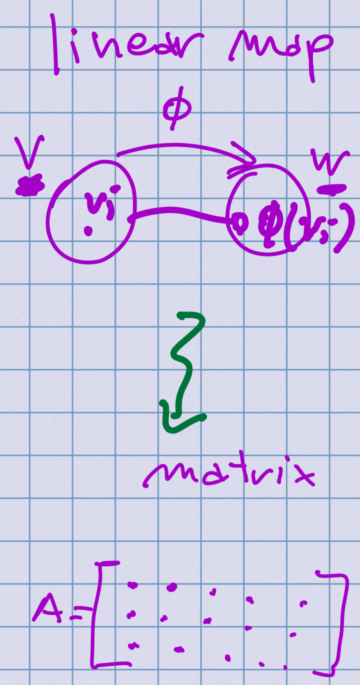


$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

→ same linear map

→ but, matrix representation depends on chosen bases.

What happens in general?



Choose bases

$\{v_1, \ldots, v_n\}$ of V

$\{w_1, \ldots, w_m\}$ of W

Apply ϕ to elements of basis of V

$$\{\phi(v_1), \ldots, \phi(v_n)\} \subseteq W$$

Decompose each as LC of basis of W

$$\phi(v_j) = \sum_i \alpha_{ij}\, w_i$$

Put coeffs α_{ij} in matrix

$$\begin{array}{cc} j & \\ \downarrow & \\ A = \begin{bmatrix} \alpha_{ij} \end{bmatrix} & \leftarrow i \end{array}$$

THE MATRICES **ARE** DIFFERENT

BUT...

MOST "IMPORTANT" PROPERTIES/QUANTITIES ARE IN FACT **BASIS INDEPENDENT**

FOR INSTANCE

$$\begin{tikzcd} O \arrow[r, bend left, "\phi"] & O \arrow[l, bend left, "\phi^{-1}?"] \end{tikzcd}$$

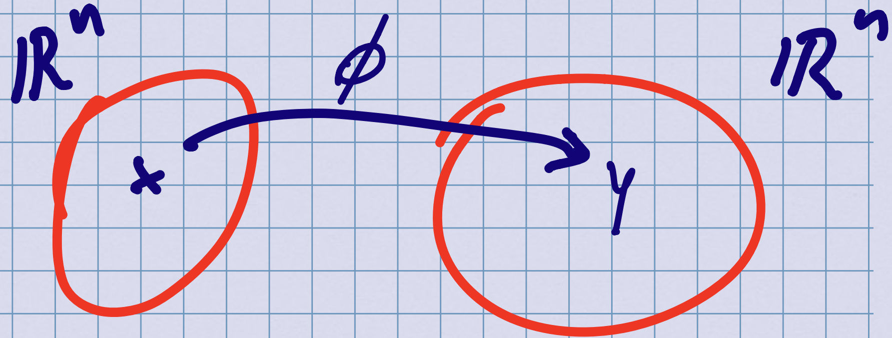
- INVERTIBILITY
- $\dim N(\phi)$
- $\dim C(\phi)$
- rank

+ potentially others (depending on what bases are allowable)

This lecture: the determinant of a square matrix

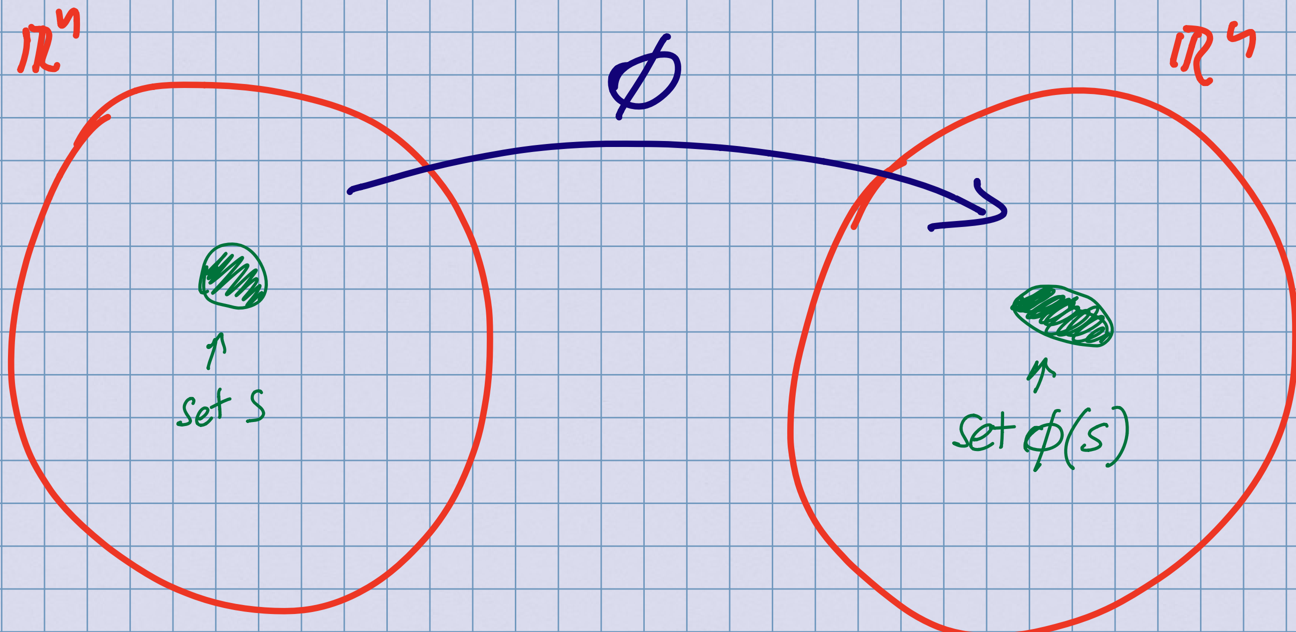
$A \in \mathbb{R}^{n \times n}$ $n = m$.

Motivation



- Want to understand how the volume of a set changes under ϕ

$$y = Ax$$
$$y = \phi(x)$$



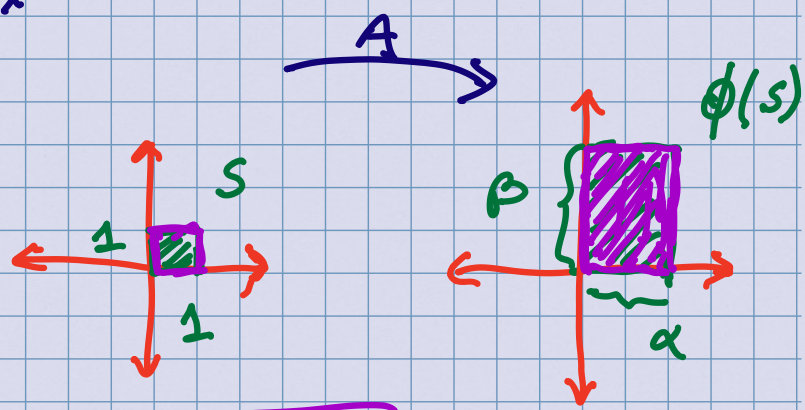
Q: How do Volume($\phi(S)$) and
Volume(S) compare?

Take this as our definition

Def: The determinant of $A \in \mathbb{R}^{n \times n}$ is
the scaling factor between
vol(S) and vol($\phi_A(S)$)
(taking into account handedness)

Ex 1: diagonal matrix

$(\alpha > 0, \beta > 0)$

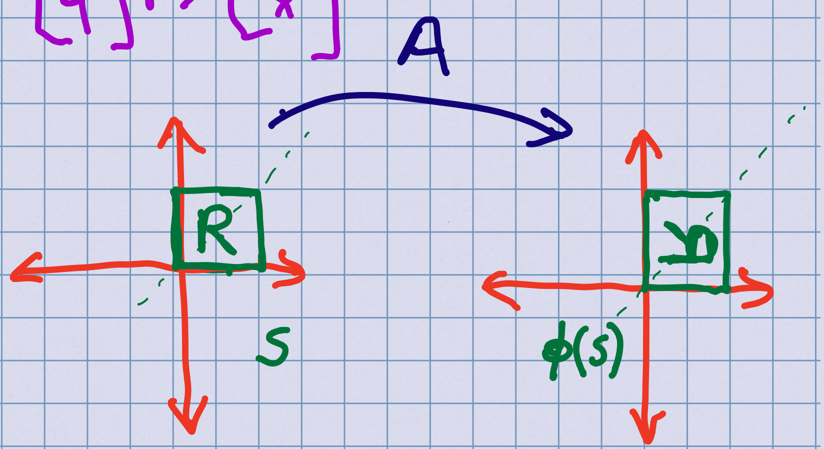


$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \alpha x \\ \beta y \end{bmatrix}$$

$$\boxed{\det A = \alpha\beta}$$

Ex 2: reflection $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$



$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

"exchange x and y"

$$\boxed{\det(A) = -1}$$

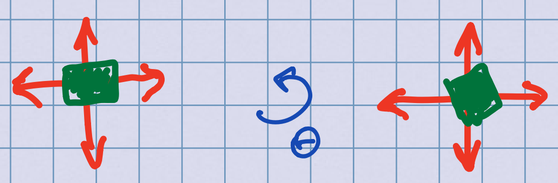
same volume, but different handedness (reflection)

Ex 3/4

$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+\alpha y \\ y \end{bmatrix}$

rotations

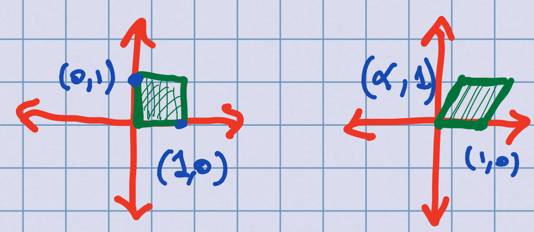
$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos(\theta) \end{bmatrix}$$



$$\boxed{\det A = 1}$$

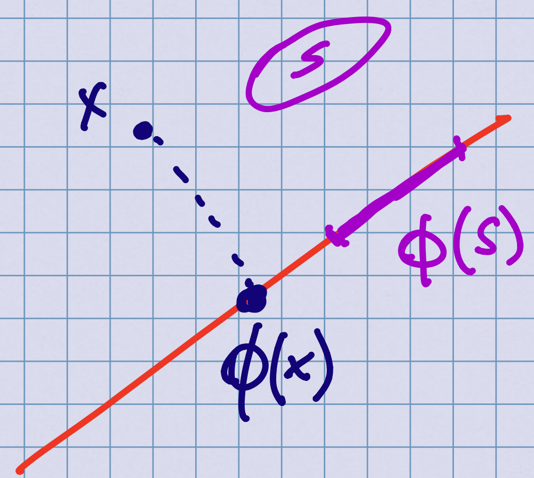
Shears

$$A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

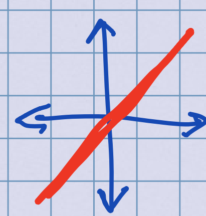


$$\boxed{\det A = 1}$$

Ex 5: PROJECTION ONTO A LINE



e.g.

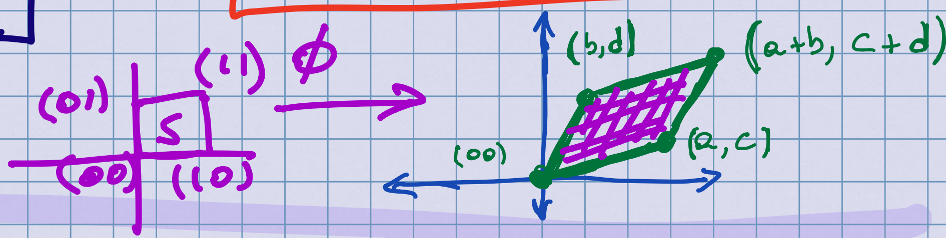


$$A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\det A = 0$$

General (2x2) case

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \boxed{\det A = ad - bc}$$



(Recall 2x2 inverse):

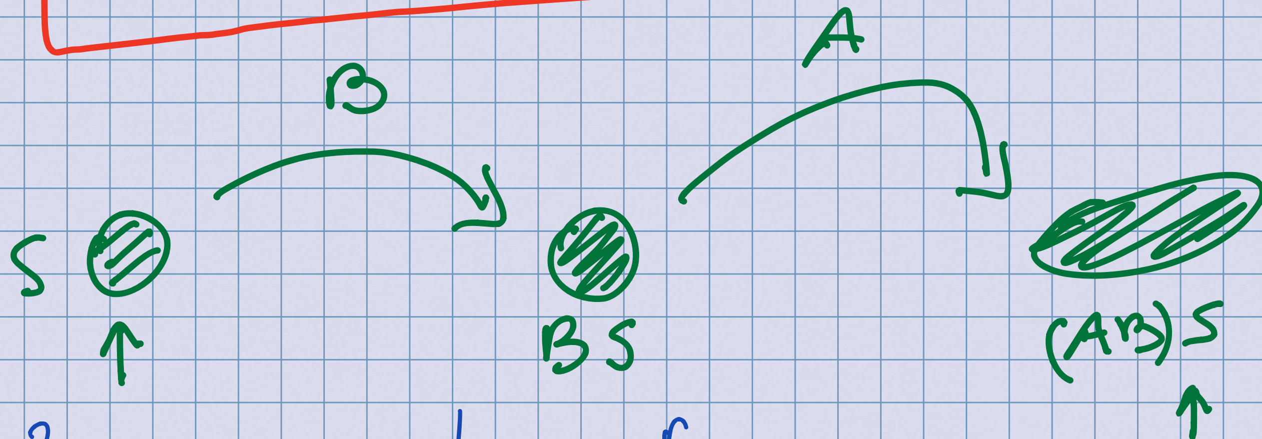
$$A^{-1} = \frac{1}{\underbrace{ad-bc}_{\det A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\boxed{A \text{ invertible iff } \det(A) \neq 0}$$

Properties

Let A, B be square matrices.

$$\boxed{\det(AB) = \det(A)\det(B)}$$



why? composition of maps

it'd be nice to write a matrix as a product of "simpler" matrices...

In general,

Gaussian elimination

Formula:

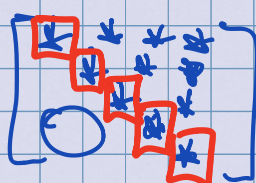
$$\underline{\det(A)} = (\pm)\cdot\left(\begin{array}{c}\text{product of}\\ \text{pivots}\\ \text{in } \underline{REF(A)}\end{array}\right)$$

depends on # of row exchanges (even/odd)

why?

$$EA = R$$ ← Gaussian Elimination

R.

$$\underbrace{(E_1 \cdots E_k)}_{\substack{\text{product of}\\ \text{elementary}\\ \text{transformations}}} A =$$  upper Triangular

det R = product of pivots (triangular)

transformations:

adding rows: det = 1
swaps : det = −1

$$\Rightarrow \det(A) = \frac{\det(R)}{\prod_i \det(E_i)}$$

More properties : $A \in \mathbb{R}^{n \times n}$

A singular (not invertible)

$\Updownarrow$

rows/columns linearly dependent

$\Updownarrow$

$$\boxed{\det A = 0}$$

$\Updownarrow$

$\operatorname{rank}(A) < n$

TRANSPOSE:

$$\boxed{\det A = \det A^T}$$

BLOCK TRIANGULAR

B, D square

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

$$\Rightarrow \boxed{\det A = \det(B) \cdot \det(D)}$$

CAVEAT

DETERMINANTS ARE GREAT, BUT THEY SHOULD ALMOST NEVER BE USED IN NUMERICAL WORK.

WE'LL LEARN MUCH BETTER WAYS LATER IN THE COURSE (eigenvalues, singular values)