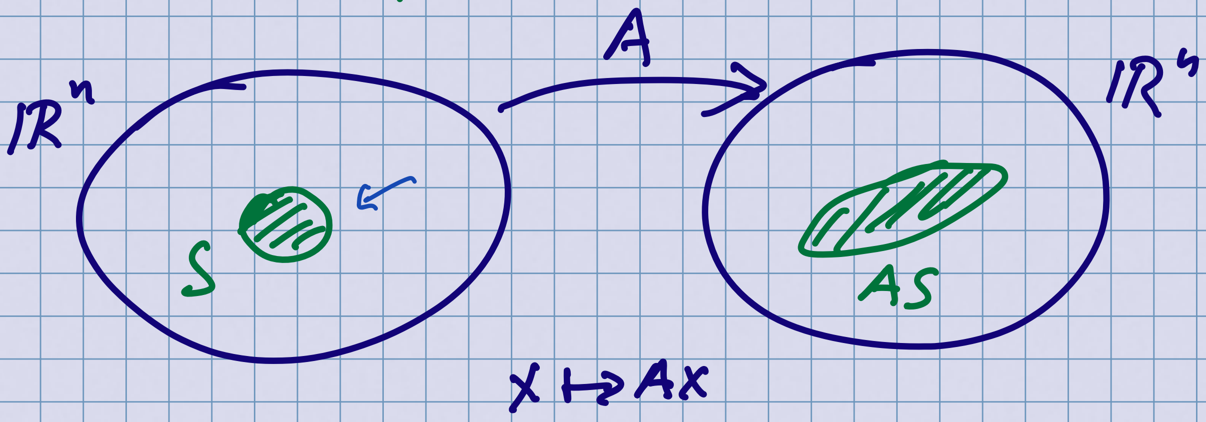
Lecture 11

The Matrix Inverse Existence and Projections.

- PSET 2 OUT!
  Due Mon Oct 5
- Midterm : Wed Oct 7
  (no lecture that day)
  (up to today's lecture)
- Final : Friday Dec 18

Determinants:

Recall: Given a square matrix $A \in \mathbb{R}^{n \times n}$ the determinant was defined as the ratio $\boxed{\dfrac{\mathrm{vol}(AS)}{\mathrm{vol}(S)}}$ (for any set S).



Example:

(2x2) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \boxed{\det A = ad - bc}$

(product) $\boxed{\det(AB) = \det(A)\det(B)}$

(diagonal)

$$\boxed{\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det(A)\det(C)}$$

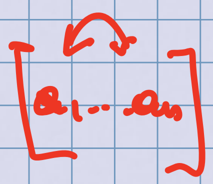
COMPUTATION:

$$\underline{\det A} = (-1)^{\#\text{row exchanges}} \left(\text{product of pivots } \underline{\text{in REF}} \right)$$

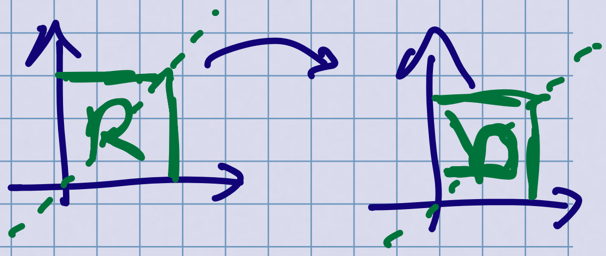
Great. Useful. But not too insightful ...

Today, a few more properties to better understand determinants ...

- ROW / COLUMN EXCHANGES



Recall this example



$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(x,y) \longmapsto (y,x)$$

swapping any two rows/columns is "like reflecting on a mirror"

$\Longrightarrow$ determinant <u>changes sign</u>

$$\det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \qquad\qquad \det\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} =$$

$$\det\begin{bmatrix} 0 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 6 & 7 \end{bmatrix} =$$

Multilinearity :

$\alpha, \beta \in \mathbb{R}$

$$\det \begin{bmatrix} \alpha v + \beta w \\ M \end{bmatrix} = \alpha \cdot \det \begin{bmatrix} v \\ M \end{bmatrix} + \beta \det \begin{bmatrix} w \\ M \end{bmatrix}$$

Linear in each row / column separately

NOT TRUE : $\det(A+B) = \det(A) + \det(B)$

In general, $\det(A)$ is

a multilinear polynomial

in the coefficients a_{ij}

More specifically:

"big formula" in the notes p. 53

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & & a_{nn} \end{bmatrix} =$$

$$= \sum_{\substack{\text{permutation} \\ \sigma \text{ of } \{1,\dots,n\}}} (-1)^{\operatorname{sgn}\sigma} \underbrace{a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}}$$

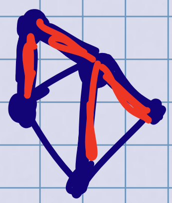
$\leftarrow$ # of pairwise exchanges

$\boxed{n! \text{ terms}}$

Many applications:

✱ later in the course

- ✱ Computing Volumes (e.g. ellipsoids)
- ✱ Multivariate Gaussian distributions
- Multidimensional Integrals
- Combinatorics (e.g. counting spanning trees)



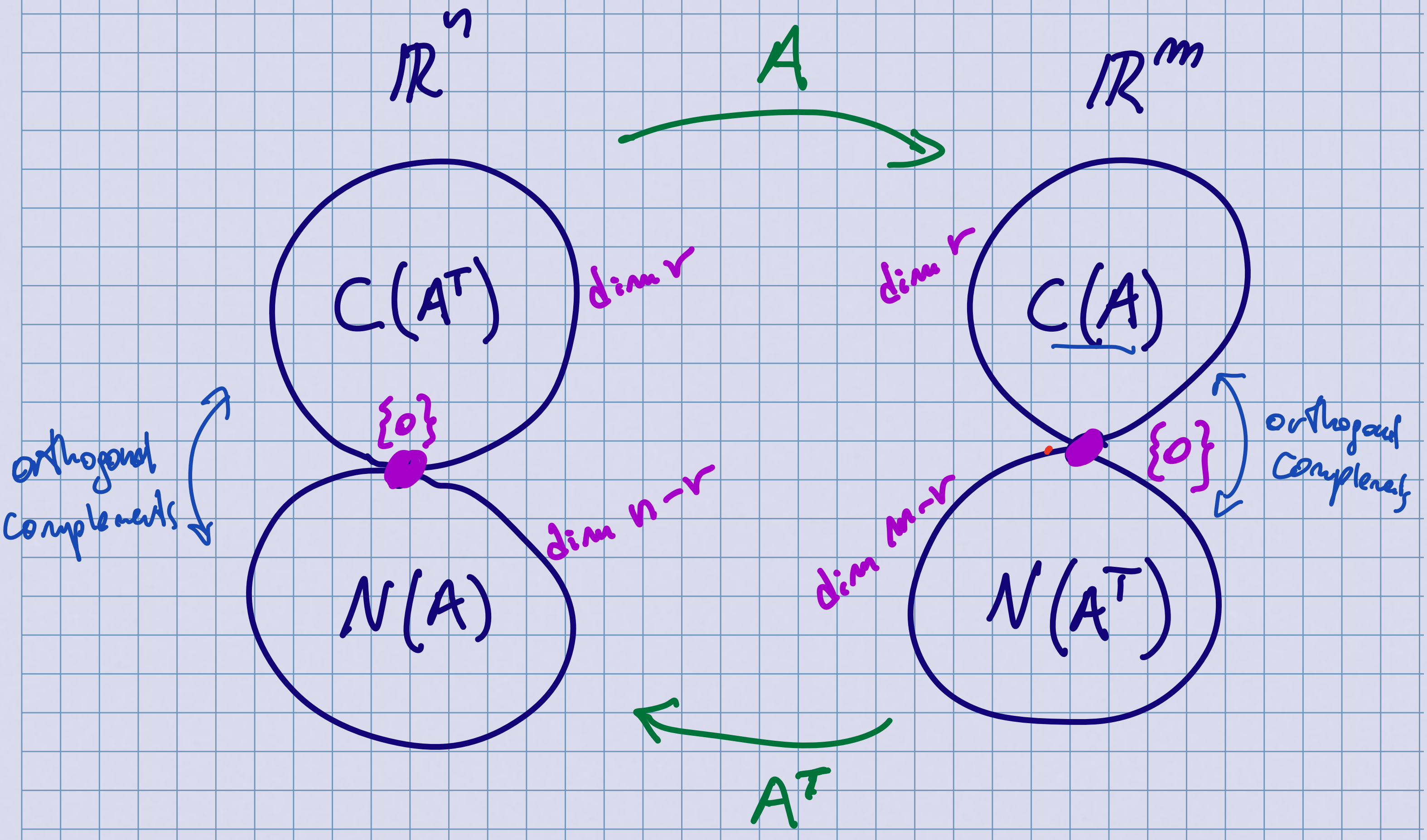
Enough about determinants... (for now)

BACK TO LINEAR MAPS!

Recall this picture:

(end of Lecture 8)

$A \in \mathbb{R}^{m \times n}$



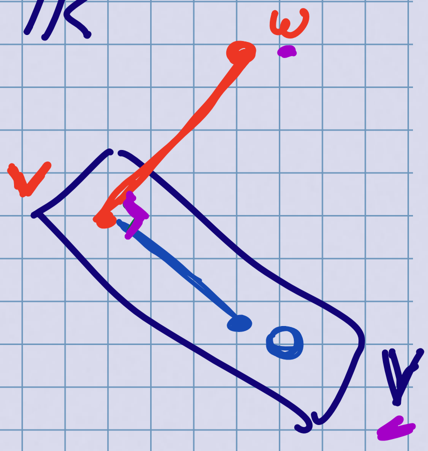
Q: How to decompose (project) a given vector?

Q: What does this say about (left/right) inverses?

Projections:

Given a subspace $\mathbb{V} \subseteq \mathbb{R}^n$

How to project a point w?



$$v = \text{proj}_{\mathbb{V}} w$$

Defining properties:

(a) $v \in \mathbb{V}$

(b) $(\underline{w - v}) \perp \mathbb{V}$

How to compute v?

Easy, if we have an orthonormal basis

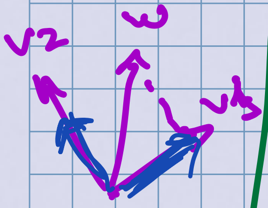
(recall Gram-Schmidt)

$$\mathbb{V} = \text{span}\{v_1, \dots, v_k\}$$

$\|v_i\| = 1$

$v_i^T v_j = 0$ $(i \neq j)$

To compute the projection $v = \text{proj}_{\mathbb{V}} w$



$$\boxed{\text{proj}_{\mathbb{V}} w = \sum_{i=1}^{k} \underbrace{(w \cdot v_i)}_{\in \mathbb{R}} v_i}$$

Need to check:

① $v \in \mathbb{V}$ ✓

② $(w-v) \perp \mathbb{V}$ ✓

Recall 

How to write this in matrix form?

$$\sum_{i=1}^{k} \overbrace{(w \cdot v_i)}^{\alpha} v_i = \sum_{i=1}^{k} v_i (v_i^T w) = Pw$$

$$\boxed{P = \sum_{i=1}^{k} v_i v_i^T}$$

Q: rank P?

Let's write it a bit nicer . . .

If $\{v_1 \dots v_k\}$ is orthonormal, then defining

$$V = \left.\underbrace{\begin{bmatrix} v_1 \cdots v_k \\ \\ \end{bmatrix}}_{k}\right\}n \in \mathbb{R}^{n \times k}$$

we have $\boxed{V^T V = I_k}$

Then, the projection matrix is

$$\boxed{P = VV^T}$$

$\left(\begin{array}{c}\text{check!} \\ P^2 = P\end{array}\right)$

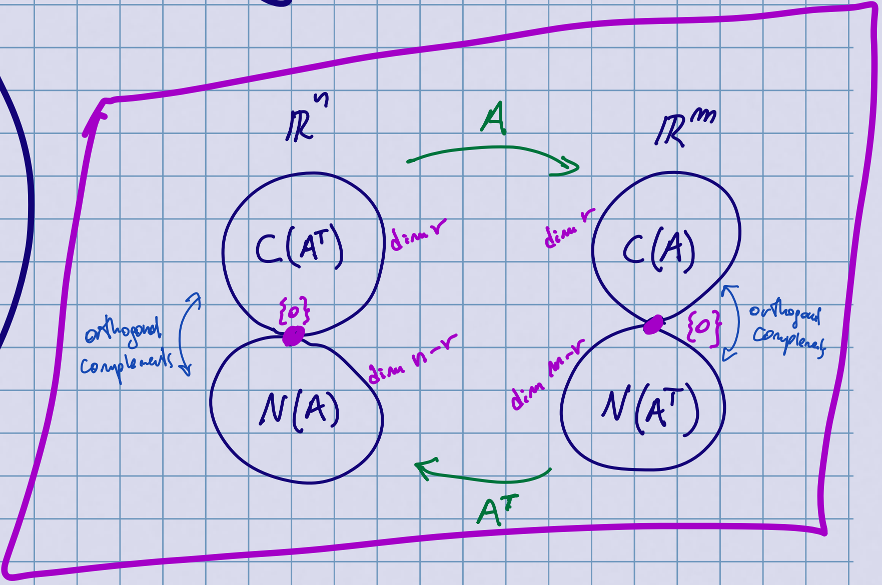
Let's verify properties: $v = Pw$

① $v \in \mathbb{V}$ easy! $Pw \in C(V)$

② $(w-v) \perp \mathbb{V}$ easy! $\underset{\substack{\uparrow \\ \text{arbitrary } u \in \mathbb{V}}}{u}^T (w-v) = u^T P \underbrace{(I-P)}_{0} w = 0$

Great. But, what if I don't want to compute an orthogonal basis . . .

(say, if I want to decompose $x \in \mathbb{R}^n$ into $\begin{cases} N(A) \\ C(A^T) \end{cases}$)

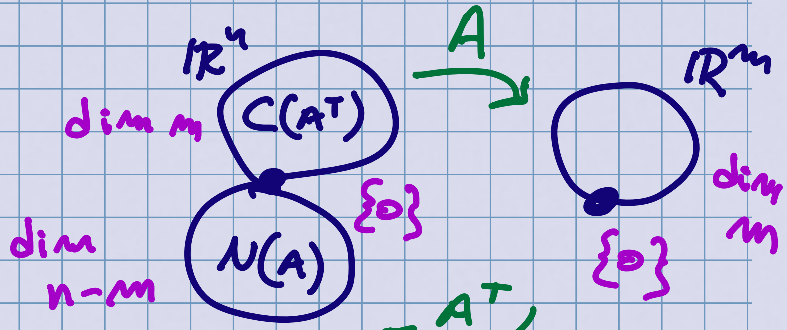


Assumption (for now): $\boxed{\operatorname{rank}(A) = m}$

"A is full row rank"

$$x = x_1 + x_2$$

$$\begin{cases} x_1 \in N(A) \\ x_2 \in C(A^T) \\ x_1 \perp x_2 \end{cases}$$



$$A = \begin{bmatrix} * & 0 \end{bmatrix}$$

(in right basis)

$$x_2 \in C(A^T) \iff x_2 = A^T z \quad \text{for some } z$$

$$x_1 = x - x_2 = x - A^T z \quad \in N(A)$$

$$\Rightarrow \quad A\,(x - A^T z) = 0 \qquad \boxed{Ax = \underbrace{AA^T}_{\text{invertible!}} z}$$

(m×m) rank m

$$\Rightarrow \quad z = (AA^T)^{-1} A x$$

$$\Rightarrow \boxed{\begin{cases} x_1 = x - A^T (AA^T)^{-1} A\, x \\ x_2 = A^T (AA^T)^{-1} A\, x \end{cases}}$$

Nicer in Matrix Form.

$$\begin{cases} P = A^T (AA^T)^{-1} A & \left[\text{orthogonal proj onto } C(A^T)\right] \\ Q = I - A^T (AA^T)^{-1} A & \left[\text{ortlogonal projection onto } N(A)\right] \end{cases}$$

Check!

$$P^2 = P$$
$$Q^2 = Q$$

(projections)

$$P + Q = I$$

(complement)

$$AQ = 0$$

Left / right inverses:

(recall Lecture 8)

a Right inverse of A: (not unique!)

$$\boxed{\underbrace{A^T(AA^T)^{-1}}_{n\times m}}$$

a Left inverse of A^T: (not unique!)

$$(AA^T)^{-1}A$$

"Dual" picture

$A \in \mathbb{R}^{m\times n}$

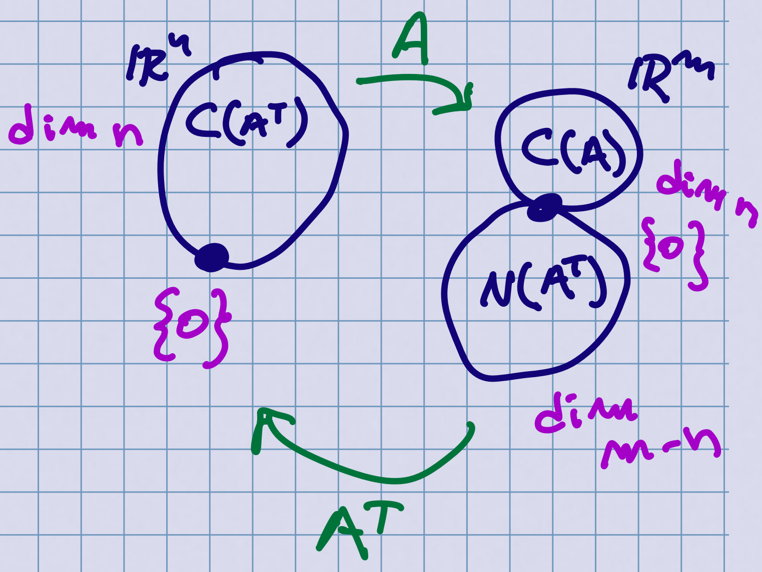
if we assume

$$\boxed{\text{rank}(A) = n}$$

"A is full column rank"

$$\Rightarrow \quad \underbrace{A^TA}_{n\times n} \quad \text{invertible}$$

rank n



$$A = \begin{bmatrix} * \\ 0 \end{bmatrix}$$

(in right basis)

For instance:

- "a" Left inverse of A:

  (not unique)

$$(A^TA)^{-1}A^T$$

...

"The" Inverse

- If A has both a right and a left inverse, they are equal: $BA = AC = I \Rightarrow B = C$

$$B = BI = BAC = IC = C$$

- The inverse, if it exists, is unique

All equivalent !

SQUARE

$\boxed{A \in \mathbb{R}^{n\times n}}$

A nonsingular (invertible)

$$\Updownarrow$$

rows linearly independent

$$\Updownarrow$$

columns linearly independent

$$\Updownarrow$$

$$\boxed{\det A \neq 0}$$

$$\Updownarrow$$

$$\operatorname{rank}(A) = n$$

$$\Updownarrow$$

$$N(A) = \{0\}$$

$$\Updownarrow$$

$$C(A) = \mathbb{R}^n$$

$$\Updownarrow$$

$Ax=b$ always has a unique sol.

Computation

Gaussian elimination!

Apply to

$$[A \quad I] \overset{\text{G.E.}}{\rightsquigarrow} \overbrace{[I \quad B]}^{\text{RREF}}$$

Then, $\boxed{B = A^{-1}}$