Lecture 9

ORTHOGONALITY AND GRAM-SCHMIDT

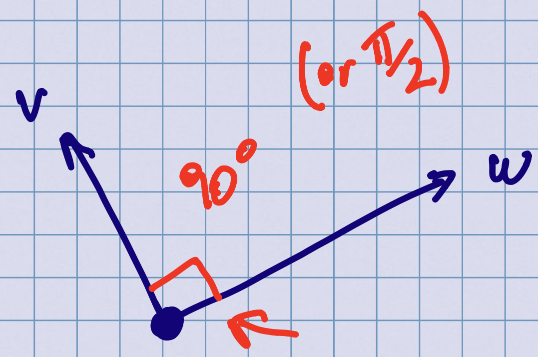
Last time:

MATRIX RANK

- (column rank) max # of LI columns
- (row rank) max # of LI rows
- (Gaussian elim.) # of pivots
- (factorization) $\underbrace{A}_{n\times m} = \underbrace{B}_{n\times k}\underbrace{C}_{k\times m}$

smallest dimension k

ORTHOGONALITY:



(PLANE GEOMETRY).

- TWO VECTORS ARE ORTHOGONAL if THE ANGLE BETWEEN THEM is 90°. (RIGHT ANGLE)

(ALSO "PERPENDICULAR")

↳ **Def**: VECTORS $\underline{v}$ AND $\underline{w}$ are ORTHOGONAL if

$$\boxed{v \cdot w = 0}$$

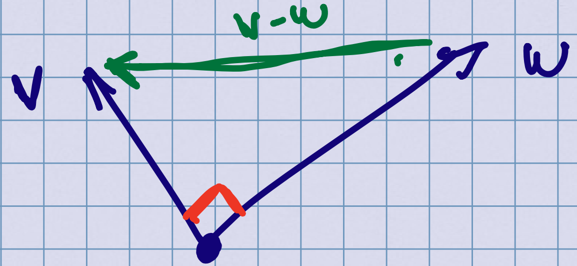
(Denote as $v \perp w$)

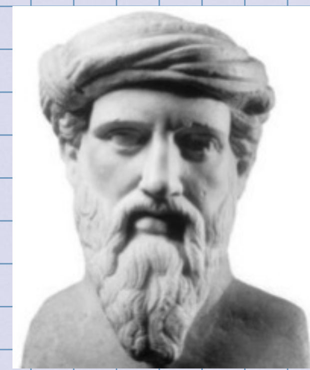
(dot product or inner product)

Recall: $v \cdot w = v^T w = \boxed{\sum v_i w_i}$

$w \cdot v = w^T v =$

WHY ORTHOGONALITY?





PYTHAGORAS

$$\hookrightarrow \|v-w\|^2 = \|v\|^2 + \|w\|^2$$

or

$$\|v+w\|^2 = \|v\|^2 + \|w\|^2$$

$$\boxed{\begin{aligned}\|v\|^2 &= v \cdot v \\ &= \sum v_i^2\end{aligned}}$$

<u>Pf</u>: $\|v+w\|^2 = (v+w)\cdot(v+w)$

$$= v\cdot v + \underbrace{v\cdot w}_{0} + \underbrace{w\cdot v}_{0} + w\cdot w = \|v\|^2 + \|w\|^2$$

USEFUL!

PAIRWISE ORTHOGONAL

$i \neq j \Rightarrow v_i \cdot v_j = 0$

NORM OF SUMS:

Eg, if $\{v_1, \dots, v_k\}$ are pairwise orthogonal

Then

$$\|\sum \lambda_i v_i\|^2 = \sum \lambda_i^2 \|v_i\|^2$$

LINEAR INDEPENDENCE:

$\{v_1 \dots v_k\}$

PAIRWISE ORTHOGONAL VECTORS ARE LI.

Why? $$(\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_k v_k) = 0$$ zero vector

$$v_1 \cdot (\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_k v_k) = 0$$ number zero

$$\lambda_1 (v_1 \cdot v_1) + \lambda_2 \underbrace{(\cancel{v_1 \cdot v_2})}_{0} + \dots + \lambda_k \underbrace{(\cancel{v_1 \cdot v_k})}_{0} = 0$$

$$\Rightarrow \lambda_1 = 0$$

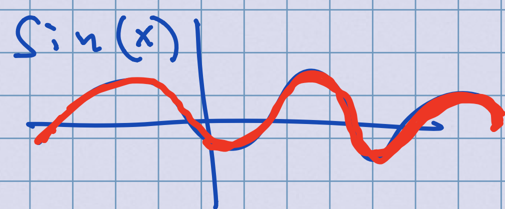
WHY SHOULD WE CARE?

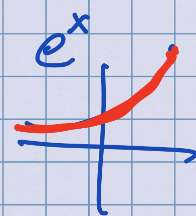
(OTHER THAN BECAUSE IT'S BEAUTIFUL AND COOL)

ENABLES US TO EXTEND GEOMETRIC IDEAS TO MANY OTHER OBJECTS (AS LONG AS THEY FORM A VECTOR SPACE)

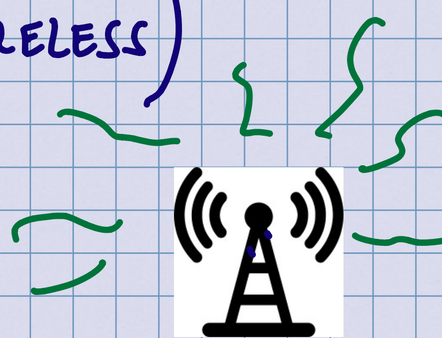
Examples:

- POLYNOMIALS $P(x) = x^n + \cdots + a_1 x + a_0$
- FUNCTIONS

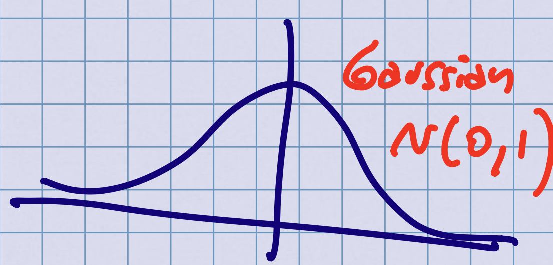




- SIGNALS (AUDIO, VIDEO, WIRELESS)



- RANDOM VARIABLES



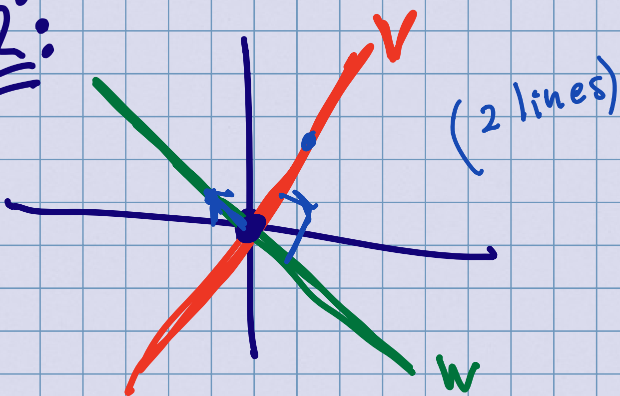
(and many more!)

ORTHOGONALITY OF SUBSPACES

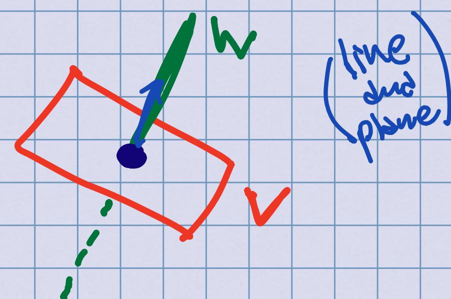
Two subspaces $V, W \subseteq \mathbb{R}^n$ are orthogonal

if $\boxed{v \cdot w = 0}$ for every $\begin{cases} v \in V \\ w \in W \end{cases}$

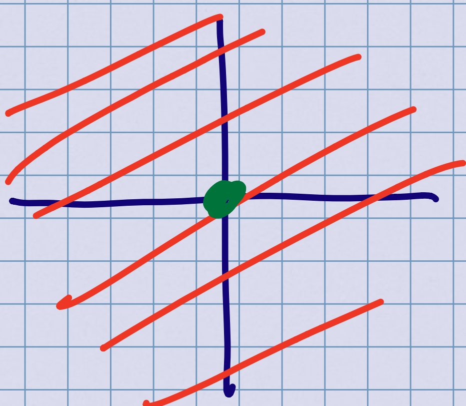
$\mathbb{R}^2$:



$\mathbb{R}^3$:



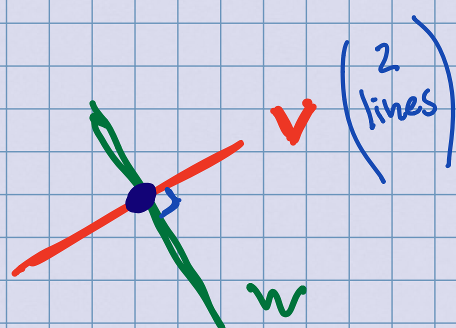
$\mathbb{R}^2$:



$V = \mathbb{R}^2$

$W = \{0\}$

$\mathbb{R}^3$:



(not the only possibilities!)

Q: If V and W are orthogonal, what can we say about their dimensions?

dim(V)
dim(W)

E.g., can we have two orthogonal planes in $\mathbb{R}^3$?

if V, W are orthogonal,

Can they intersect?

$\Rightarrow$ NO!

$\exists\, p \in V$, $p \in W$

$p \cdot p = 0$ $= \|p\|^2 = 0$

$\Rightarrow \boxed{p = 0}$

$$\dim(V) + \dim(W) \leq n$$

ORTHOGONAL COMPLEMENT

Def: Given $V \subseteq \mathbb{R}^n$, its orthogonal complement

$$V^{\perp} = \{ w \in \mathbb{R}^n : \quad w \cdot v = 0 \quad \forall\, v \in V \}$$

"vectors that are orthogonal to all $v \in V$"

Q: Is $V^{\perp}$ a subspace?

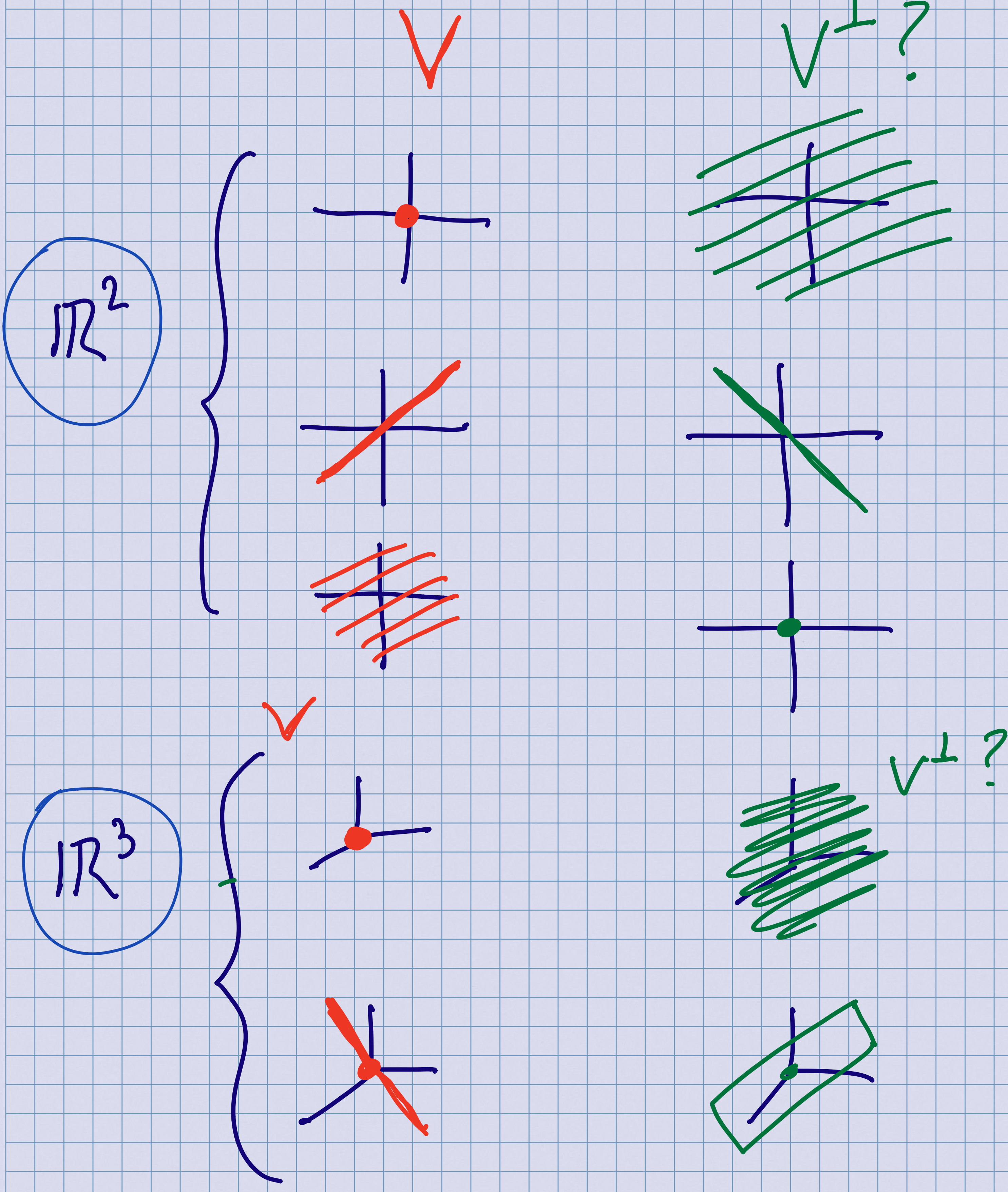
- closed under addition ✓
- " under multip. by scalars. ✓

$$w_1 \in V^{\perp} \quad \rightsquigarrow \quad w_1 \cdot v = 0 \quad \forall v \in V$$

$$w_2 \in V^{\perp} \qquad\qquad w_2 \cdot v = 0 \quad \forall v \in V$$

$$w_1 + w_2 \in V^{\perp} \Leftarrow (w_1 + w_2) \cdot v = 0 \quad \forall v \in V$$

Examples: (recall from before)



Hmmmmm...

Q: What can we say about

$$\dim(V^\perp)\ ?$$

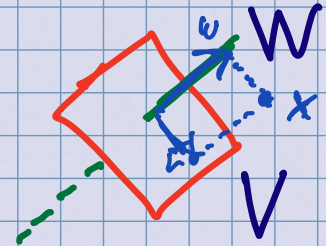
$$\boxed{\dim(V) + \dim(V^\perp) = n}$$

Q: What can we say about

$$(V^\perp)^\perp\ ?$$

$$(V^\perp)^\perp = V$$

Def: If V, W are orthogonal complements of each other (i.e., $V = W^{\perp}$, $W = V^{\perp}$) they form an orthogonal decomposition



Then, every vector x in the space can be written uniquely as

$$x = v + w$$

where $\begin{cases} v \in V \\ w \in W \end{cases}$ and $v \cdot w = 0$

Q: where do these decompositions come from?

Hold on...

Even before:

Q: Where do orthogonal vectors come from??

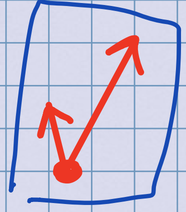
We can make them!

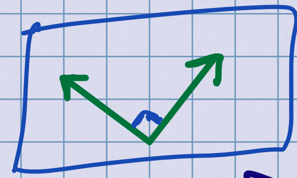
Gram-Schmidt, Inc.

GRAM-SCHMIDT:

a procedure (algorithm)

to orthogonalize a set of LI vectors





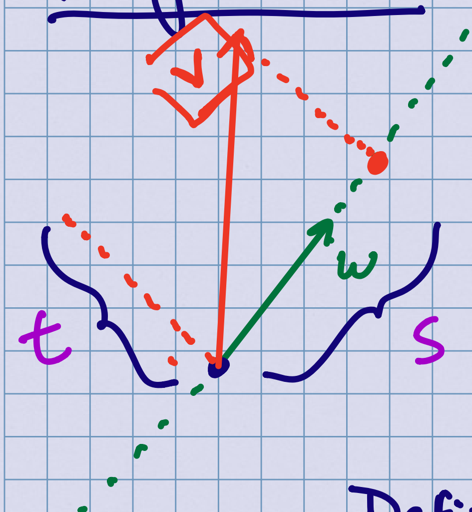
$\{v_1, \dots, v_n\}$ linearly independent

ALGORITHM

$\{w_1, \dots w_n\}$ orthonormal vectors

$$\begin{cases} \|w_i\| = 1 & \\ w_i \cdot w_j = 0 & i \neq j \end{cases}$$

Projection: Given w, with $\|w\| = 1$.



The projection of v onto w is

$$\text{proj}_w v = (\underbrace{v \cdot w}_{\alpha \in \mathbb{R}})\, w$$

Define:

$$\begin{cases} s = \text{proj}_w v \\ t = v - \text{proj}_w v \end{cases}$$

Claim: decomposition $v = s + t$, $s \cdot t = 0$

$$s \cdot t = (\alpha w) \cdot (v - \alpha w) = \alpha \overbrace{(v \cdot w)}^{\alpha} - \alpha^2 \overbrace{w \cdot w}^{1} = 0.$$

Input: $\{v_1, \ldots, v_n\}$ LI vectors

Idea: At every step, subtract the projection onto all previously found vectors.

$w_1 = \frac{v_1}{\|v_1\|}$

$\boxed{v_1:}$ $w_1 = \text{normalize}(v_1)$

$\boxed{v_2:}$ $w_2 = \text{normalize}\left(v_2 - \text{proj}_{w_1} v_2\right)$

$v_3:$ $w_3 = \text{normalize}\left(v_3 - \text{proj}_{w_1} v_3 - \text{proj}_{w_2} v_3\right)$

$\vdots$

$v_n:$ $w_n = \text{normalize}\left(v_n - \sum_{k=1}^{n-1} \text{proj}_{w_k} v_n\right)$

Claim:

- $\{w_1, \ldots, w_n\}$ are orthonormal
- $\text{span}\{w_i\} = \text{span}\{v_i\}$

i.e, an "ORTHONORMAL BASIS"

A natural source of orthogonal decompositions:

MATRICES!

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$$

$$x \longmapsto Ax$$

Given $\boxed{A \in \mathbb{R}^{m \times n}}$

nullspace $\underline{N(A) \subseteq \mathbb{R}^n}$ and column space $\underline{C(A) \subseteq \mathbb{R}^m}$

Great. But, we need more...

We also have $A^T \in \mathbb{R}^{n \times m}$!

$A^T \in \mathbb{R}^{n \times m}$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, bend left, "A"] & \mathbb{R}^m \arrow[l, bend left, "A^T"] \end{tikzcd}$$

(not inverse)

where is $N(A^T)$?

$C(A^T) \subseteq \mathbb{R}^n$

$N(A^T) \subseteq \mathbb{R}^m$

Hmmm...

$\mathbb{R}^n$ =

$N(A)$

$C(A^T)$

ORTHOGONAL COMPLEMENTS

$\mathbb{R}^m$ =

$C(A)$

$N(A^T)$

Why?

They are orthogonal:

Take $v \in N(A)$, $w \in C(A^T)$.

Then $w \cdot v = w^T v = (\underbrace{A^T z}_{w = A^T z})^T v = z^T \underbrace{A v}_{0} = 0$

(same for other).

why do they fill the space?

What about dimensions?

If $r = \text{rank}(A) = \text{rank}(A^T)$

then

(rank-nullity)

$$\boxed{\begin{aligned} \dim C(A^T) &= r \\ \dim N(A) &= n-r \end{aligned}}$$

$\mathbb{R}^n$

(rank-nullity)

$$\boxed{\begin{aligned} \dim C(A) &= r \\ \dim N(A^T) &= m-r \end{aligned}}$$

$\mathbb{R}^m$

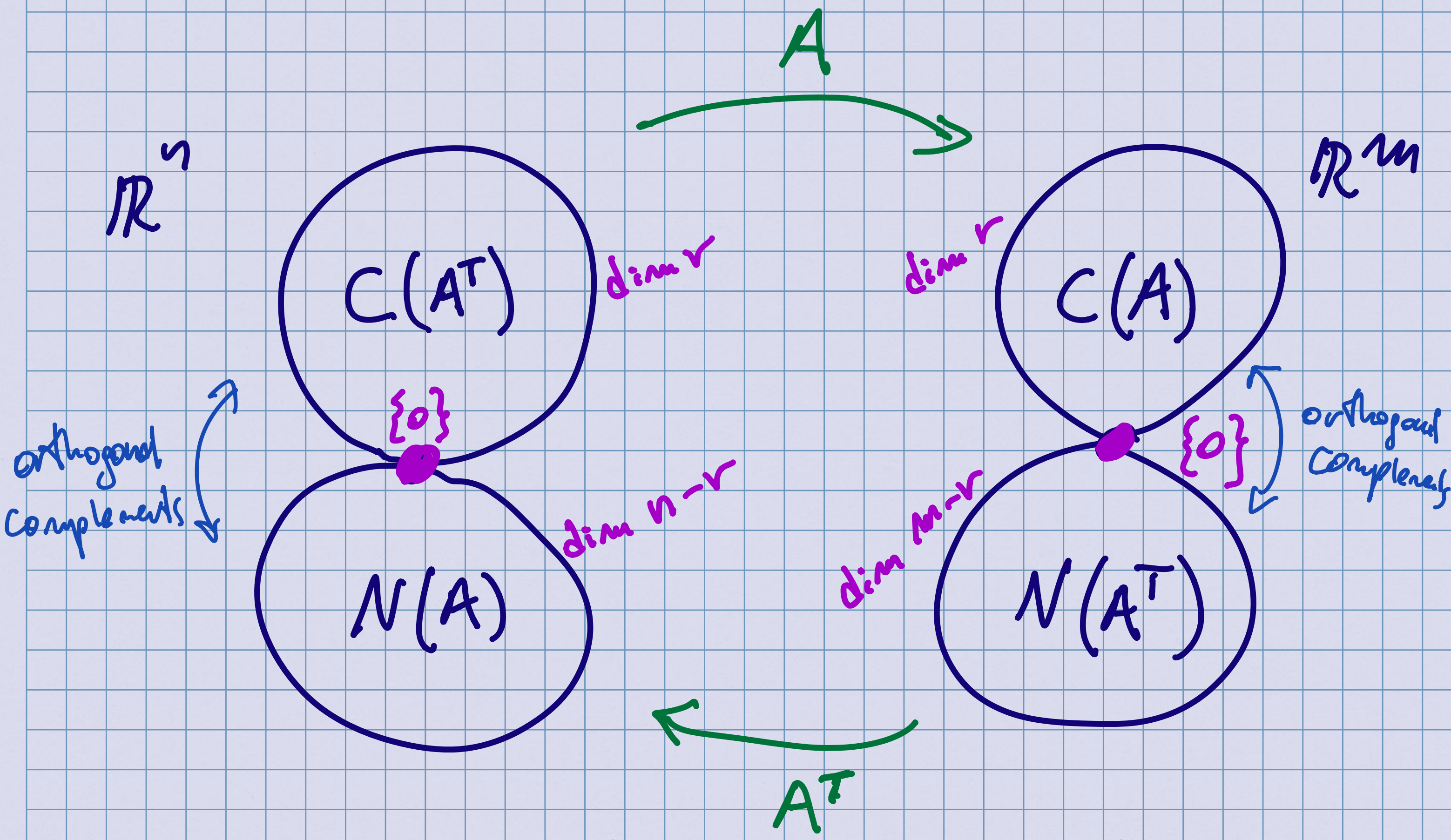
Thus, any vector $\underline{v} \in \mathbb{R}^n$ can be decomposed uniquely as

$$\boxed{\underline{v} = \underbrace{v_1}_{\in N(A)} + \underbrace{v_2}_{C(A^T)}}$$

with $\boxed{v_1 \cdot v_2 = 0}$, $\quad Av_1 = 0 \quad v_2 = A^T w$

Will allow us to understand the structure of matrices (and linear maps) in a simpler way.

THE BIG PICTURE



More, next time!