Lecture 17

Eigenvalues and Eigenvectors

Recall SVD $A \in \mathbb{R}^{m \times n}$

$$A = \underset{\substack{\uparrow \\ \text{orthogonal}}}{U}\; \underset{\substack{\uparrow \\ \text{"diagonal"}}}{\Sigma}\; \underset{\substack{\uparrow \\ \text{orthogonal}}}{V^T}$$

Nice, because it allows us to understand the structure of linear maps between two spaces: in the "right" coordinates, they are "diagonal".

TODAY: EIGENVALUES! MANY REASONS, INCLUDING:

- UNDERSTAND LINEAR MAPS FROM A SPACE TO ITSELF.

- UNDERSTAND HOW TO COMPUTE SVD.
  (or why it exists!)

We consider linear maps

from $\mathbb{R}^n$ to $\mathbb{R}^n$

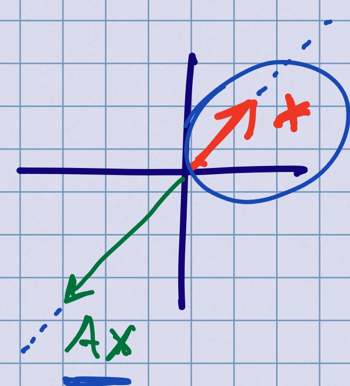
(in general, $A: V \longrightarrow V$, same vector space)

Thus, the matrix A is square.

Interested in a key property:

- Want the image of the vector x, (under $x \mapsto Ax$) to be a scalar multiple of it:

$$\overbrace{A}^{n\times n}\underbrace{\overbrace{x}^{n\times 1}}_{\text{vector}} = \underbrace{\overbrace{\lambda}^{1\times 1}}_{\text{scalar}}\ \underbrace{\overbrace{x}^{n\times 1}}_{\text{vector}}$$



$$\underbrace{A}_{}^{n\times n}\ \overbrace{x}^{n\times 1} = \overbrace{\lambda}^{1\times 1}\ \overbrace{x}^{n\times 1} \qquad x \neq 0$$

vector scalar vector

When such λ and x exist,

we'll call them λ : eigenvalue
x : eigenvector

("Eigen" is German for { self- / own / proper })

Obs: Only direction of x matters.

$Ax = \lambda x$ $\quad x$ is an eigenvector

$A(\underbrace{7x}_{y}) = \lambda(\underbrace{7x}_{y})$ $\quad 7x$ is an eigenvector

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$

Is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ an eigenvector? For which Eigenvalue?

$Ax \stackrel{?}{=} \lambda x$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 8 \\ 12 \end{bmatrix}}_{Ax} = \underbrace{4}_{\lambda} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

What about $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$? (poll) YES! $\lambda = -1$

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$Ax = (-1) \cdot x$

What about $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$? (poll) No!

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

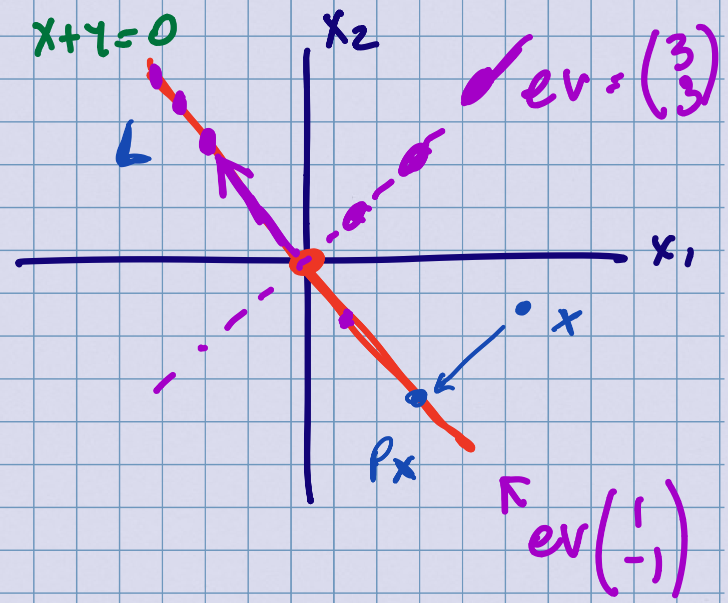
not a multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Why do we care?

Because (like SVD), it can give us a lot of insight on the structure of the matrix A (and the associated linear map).

Example: orthogonal Projection onto a line.

$$P = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$



Q: What are the eigenvalues / eigenvectors of P, and what do they tell us?

Take

$$x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow Px = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot x$$

"1 is the eigenvalue associated w/ the eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$"

Take

$$x = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \Rightarrow Px = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot x$$

"0 is the eigenvalue associated w/ the eigenvector $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$"

The eigenvectors give us <u>geometric information</u> about the <u>projection</u> P (e.g., position of the line).

How to find eigenvalues / eigenvectors?
(or, do they even exist?)

$$Ax = \lambda x \qquad (x \neq 0)$$

Rewrite as:

$A \in \mathbb{R}^{n \times n}$
$I_n \in \mathbb{R}^{n \times n}$

$$\exists x \neq 0 \qquad (A - \lambda I)\, x = 0$$

$$\Updownarrow$$

$$A - \lambda I \text{ is } \underline{\text{singular}}$$

(since it has a nonzero vector in the nullspace)

$$\Updownarrow$$

$$\boxed{\det(A - \lambda I) = 0}$$

Define $$p(\lambda) := \det(A - \lambda I)$$

TWO OBSERVATIONS:

- $p(\lambda)$ is a polynomial in λ
- $p(\lambda)$ has degree n (if $A \in \mathbb{R}^{n \times n}$).

$\boxed{\text{Def: } p(\lambda) := \det(A - \lambda I) \text{ is the characteristic polynomial of } A.}$

(sign convention) $\det(A - \lambda I)$

vs $\det(\lambda I - A)$

Example:

(from before)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$$p(\lambda) = \det(A - \lambda I)$$

$$= \det \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix}$$

$$= (1-\lambda)(2-\lambda) - 6$$

$$= 2 - 3\lambda + \lambda^2 - 6$$

$$= \boxed{\lambda^2 - 3\lambda - 4} = (\lambda - 4)(\lambda + 1)$$

Eigenvectors?

$$Ax = \lambda x \iff (A - \lambda I)x = 0$$

$\lambda = 4$ $$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} x = 0 \Rightarrow x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$\lambda = -1$ $$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} x = 0 \Rightarrow x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Lemma: λ is an eigenvalue iff λ is a root of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$

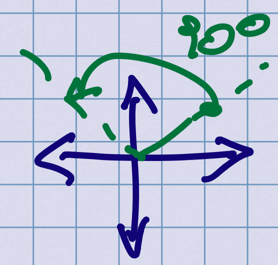
Recall that every polynomial of degree n has n roots*. (fundamental theorem of algebra)

Careful:
- May be complex (e.g. $\lambda^2 + 1$)
- Counted with multiplicity (e.g. λ^2)

$\Rightarrow$ Every matrix $A \in \mathbb{R}^{n \times n}$ has exactly n eigenvalues
- may be complex
- counted with multiplicity

Example:

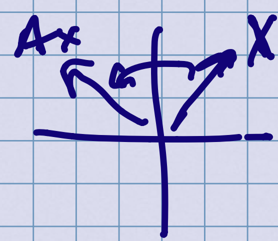
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



90° rotation

Eigenvalues??

$$Ax = \lambda x$$



$$p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1$$

$$\boxed{\lambda = i, \ \lambda = -i}$$

$$\boxed{i^2 = -1}$$

Eigenvectors?

$\lambda = i$ $\quad A - \lambda I = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \Rightarrow \boxed{x = \begin{pmatrix} i \\ 1 \end{pmatrix}}$

$\lambda = -i$ $\quad A - \lambda I = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \Rightarrow \boxed{x = \begin{pmatrix} -i \\ 1 \end{pmatrix}}$

$$A = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}$$

$\Rightarrow$ Do it! (after lecture!).

Lemma: If $v_1, \dots, v_k$ are eigenvectors associated to different eigenvalues $\lambda_1 \dots \lambda_k$, then they ($v_i$) are linearly independent.

E.g., for two: $Ax = \lambda x \quad Ay = \mu y \qquad \lambda \neq \mu$.

Want to show that x and y are LI.

$$\alpha x + \beta y = 0 \qquad \Longrightarrow \qquad \alpha = \beta = 0.$$

(A). $\alpha Ax + \beta Ay = 0$

$\boxed{\alpha \lambda x + \beta \mu y = 0}$.

(λ) $\boxed{\alpha \lambda x + \beta \lambda y = 0}$.

$$\beta \underbrace{(\lambda - \mu)}_{\neq 0} \underbrace{y}_{\neq 0} = 0 \quad \Rightarrow \quad \beta = 0$$

$$\Downarrow$$

$$\alpha = 0$$

ASSUME: that A has n linearly independent eigenvectors.

(e.g., this happens if all n eigenvalues are distinct).

We have $$A v_i = \lambda_i v_i \qquad i = 1 \ldots n$$

Then: $$A \underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{n \times n} = \underbrace{\begin{bmatrix} \lambda_1 v_1 & \cdots & \lambda_n v_n \end{bmatrix}}_{n \times n}$$

$$A \underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{T} = \underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{D}$$

$$\boxed{AT = TD}$$

Diagonalization:

By the assumption, T is invertible. (why?). Then,

$$AT = TD$$

$$\Updownarrow$$

$$A = TDT^{-1}$$

We say that A is diagonalizable.

Gives a factorization of A (different than the SVD).

Exercise: write an explicit factorization of $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$