Lecture 18

The Eigendecomposition

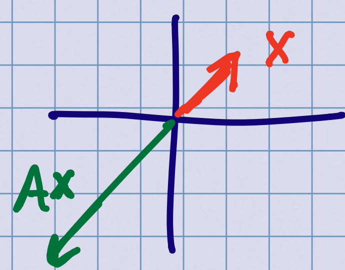
Algebraic vs. Geometric Multiplicity.

Last time:

eigenvalue equation

$$Ax = \lambda x$$

scalar
(eigenvalue)



Characteristic polynomial:

$$p(t) = \det(A - tI)$$

$$\boxed{\lambda \text{ is an eigenvalue of } A \iff \lambda \text{ is a root of } p}$$

Every matrix $A \in \mathbb{R}^{n \times n}$ has exactly n eigenvalues

- possibly complex
- counted with multiplicity

- Eigenvectors corresponding to different eigenvalues are LI.

- If all eigenvalues λ_i are distinct, then $\{v_1, \dots v_n\}$ are LI, thus

$$A \cdot \overbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}^{T} = \overbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}^{T} \overbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}^{D}$$

(diagonal)

with T invertible.

$$\Rightarrow \boxed{A = T D T^{-1}}$$

A is diagonalizable

CHARACTERISTIC POLYNOMIAL

$$p(\lambda) = \det(A - \lambda I)$$

$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots\cdots (\lambda_n - \lambda)$$

$$= p_0 + p_1 \lambda + p_2 \lambda^2 + \ldots + p_{n-1} \lambda^{n-1} + (-1)^n \lambda^n$$

DETERMINANT

$$\boxed{\det(A) = p(0) = p_0 = \lambda_1 \cdots \lambda_n}$$

(the product of eigenvalues is the determinant)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \qquad \begin{matrix} \lambda_1 = 4 \\ \lambda_2 = -1 \end{matrix}$$

TRACE : THE **TRACE** OF A SQUARE MATRIX IS

$$Tr(A) = \sum_{i=1}^{n} A_{ii}$$ (sum of diagonal elements)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

$$\boxed{Tr(A) = (-1)^n\, p_{n-1} = \lambda_1 + \cdots + \lambda_n}$$

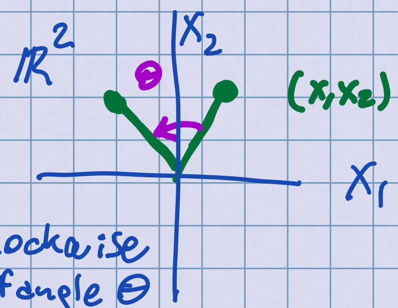
(the sum of the eigenvalues is equal to the trace)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \qquad \lambda_1 = 4 \quad \lambda_2 = -1$$

EXAMPLE :

2D ROTATIONS

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



counterclockwise rotation of angle θ

What are the eigenvalues of $\underline{R}$?

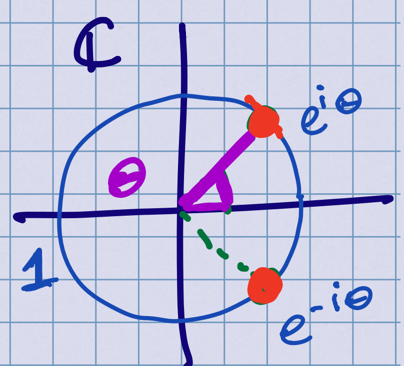
What do you expect? (Quiz!)

Let's see! $\det(R - \lambda I) = \det\begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix}$

$= (\lambda - \cos\theta)^2 + \sin^2\theta = 0$

$\Rightarrow$ $\boxed{\lambda = \cos\theta \pm i \sin\theta}$ (cartesian)

or $\boxed{\lambda = e^{\pm i\theta}}$ (polar)



"phase" gives the rotation angle!

Diagonalization:

$$\lambda_1 = e^{i\theta} \qquad A-\lambda I = \begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} \Rightarrow \quad v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\lambda_2 = e^{-i\theta} \qquad A-\lambda I = \begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix} \Rightarrow \quad v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\Rightarrow T = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad T^{-1} = \frac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$$

$$\Rightarrow \qquad A = TDT^{-1}$$

$$\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{A} = \frac{1}{2}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}}_{D} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$$

DIAGONALIZATION

$$\boxed{A = TDT^{-1}} \quad (*)$$

D : diagonal entries are eigenvalues $\quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

T : columns are eigenvectors $\quad T = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$

GREAT, <u>IF IT EXISTS</u>. WHY?

(for instance) $A^2 =$

$A^3 =$

$A^k =$

BUT, (*) DOES NOT <u>ALWAYS</u> EXIST.

:(

WHEN CAN WE DIAGONALIZE A MATRIX?

- WE ALREADY SAW A SUFFICIENT CONDITION:

  ALL N EIGENVALUES ARE DISTINCT $\left(\lambda_i \neq \lambda_j \ \forall i,j \right)$

  (since this implies eigenvectors are LI).

GREAT, BECAUSE "ALMOST ALL" MATRICES SATISFY THIS!

i.e., if you choose a matrix "generically", all eigenvalues will be different.

SO, ARE WE DONE?

NOT QUITE! THE NON-GENERIC CASE IS ALSO IMPORTANT!

WHAT COULD GO WRONG? $\quad A = TDT^{-1}$

- **STEP 1:** CHARACTERISTIC POLY

$$p(\lambda) = \det(A - \lambda I)$$

- **STEP 2:** FIND EIGENVALUES (roots of p)

$$p(\lambda) = \pm(\lambda - \lambda_1)\cdot(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

FUNDAMENTAL THM OF ALGEBRA
(possibly complex λ_i, multiplicities)
"repeated roots"

- **STEP 3:** EIGENVECTORS.

↳ if λ_i are distinct

↳ if λ_i are repeated

EXAMPLE 1:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

"good"

$$p(\lambda) = \det(A - \lambda I) = (3-\lambda)(3-\lambda)$$
$$= (3-\lambda)^2$$

$$\Rightarrow \boxed{\lambda_1 = 3 \qquad \lambda_2 = 3}$$

(twice repeated, multiplicity 2)

$\Rightarrow$ Find eigenvectors? $(A - \lambda I)x = 0$

$$A - \lambda I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} x = 0$$

choose

$$\Rightarrow \underbrace{v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{LI}$$

$$\Rightarrow A = T D T^{-1}$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\boxed{\dim N(A - \lambda I) = 2} = 2$$

EXAMPLE 2:

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

"bad"

$$p(\lambda) = \det(A - \lambda I) = (3-\lambda)(1-\lambda) + 1$$
$$= \lambda^2 - 4\lambda + 4$$
$$= (\lambda - 2)^2$$

$$\Rightarrow \boxed{\lambda_1 = 2 \qquad \lambda_2 = 2}$$

(twice repeated, multiplicity 2)

$\Rightarrow$ Find eigenvectors? $\quad (A - \lambda I)x = 0$

$$A - \lambda I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} x = 0$$

$$\Rightarrow \quad x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

only possibility

CANNOT FIND TWO LI eigenvectors

$$\boxed{\dim N(A - \lambda I) = 1} \neq 2$$

ALGEBRAIC MULTIPLICITY

vs.

GEOMETRIC MULTIPLICITY.

Consider an eigenvalue λ_0 that is repeated k times

$$p(\lambda) = \det(A - \lambda I)$$

$$= q(\lambda) \cdot \underbrace{(\lambda - \lambda_0)}_{\text{repeated factor}}{}^{k}$$

"λ_0 is a root of $p(\lambda)$ of multiplicity k".

Def: The eigenvalue λ_0 has algebraic multiplicity k.

if $p(\lambda)$ has a factor $(\lambda - \lambda_0)^k$

Recall, to compute eigenvectors,

look at $(A - \lambda I)x = 0$ (ie, the nullspace)

$\Rightarrow$ **Def**: The eigenvalue λ_0 has **geometric multiplicity** K if

$$\dim N(A - \lambda_0 I) = K.$$

Thm: A matrix is **diagonalizable** if and only if

$$\boxed{\text{alg. mult}(\lambda) = \text{geom mult}(\lambda)}$$

for **every** eigenvalue λ.

Q: What happens if A is <u>not</u> diagonalizable?

e.g., for $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

One can put A in a "block diagonal" form. (<u>Jordan</u> form).

(we won't be doing this in this course).

<u>Idea:</u> $A = T\,J\,T^{-1} \in \mathbb{R}^{n\times n}$

k distinct eigvals

$$p(\lambda) = \pm(\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2}\cdots(\lambda-\lambda_k)^{n_k}$$

$$n_1+n_2+\cdots+n_k = n$$

$$J = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & J_k \end{bmatrix}$$
$$\underbrace{\quad}_{n_1}\ \underbrace{\quad}_{n_2}\ \underbrace{\quad}_{n_k}$$

J_1 assoc. to λ_1

J_2 assoc to λ_2

J_k assoc to λ_k

EXAMPLE : 3D ROTATION.

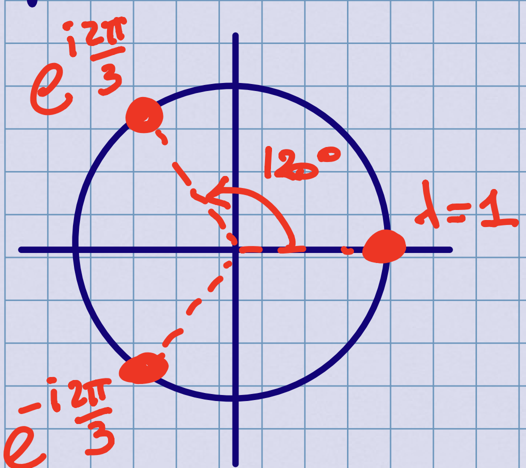
$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Q: Is this a 3D rotation?
Around which axis?
What is the angle?

Q: What do you expect the eigenvalues of R to be?

How to read the desired information?

$$p(\lambda) = \det(A - \lambda I) = 1 - \lambda^3$$



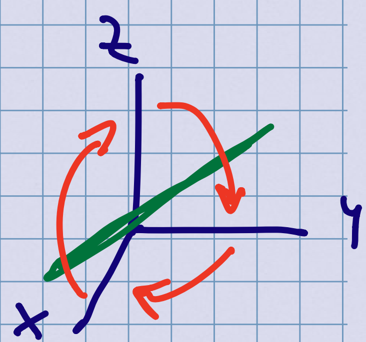
$$= \underbrace{(1-\lambda)}_{\substack{\uparrow \\ \text{1 real eig} \\ (\lambda = 1)}} \underbrace{(\lambda^2 + \lambda + 1)}_{\substack{\uparrow \\ \text{2 complex} \\ (\lambda = e^{\pm i\frac{2\pi}{3}})}}$$

- One real eigenvalue $(\lambda = 1)$.

Associated eigenvector? $\Longrightarrow$ $V = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$RV = 1 . V$

this is the axis of rotation!



- Two complex eigs: $\lambda = \begin{cases} e^{\frac{2\pi}{3} i} \\ e^{-\frac{2\pi}{3} i} \end{cases}$

$\left(\frac{2\pi}{3} = 120^\circ\right)$

This is the angle of rotation!

$\Rightarrow$ $\boxed{120^\circ \text{ around } (1,1,1).}$