Lecture 26

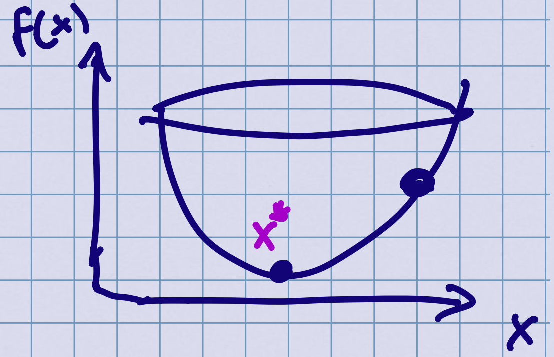
Convex sets

and Convex functions

* PSET 4 is out! $\boxed{\text{Due TODAY 9PM.}}$

* MIDTERM $\boxed{\text{FRIDAY 11/13}}$

QUADRATIC PROGRAMMING



- UNCONSTRAINED.

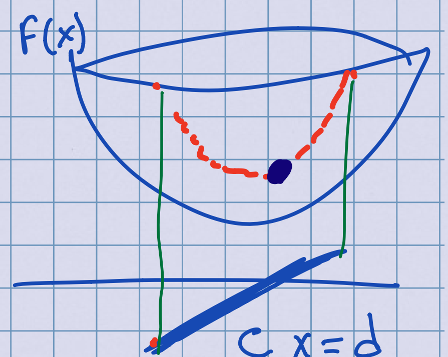
$$\min_x \; \overbrace{\frac{1}{2} x^T A x}^{\text{quadratic}} - \overbrace{b^T x}^{\text{linear}}$$

symmetric matrix

$\nabla f = 0$

(opt condition : $Ax = b$)

- EQUALITY CONSTRAINED



$$\min_x \; \frac{1}{2} x^T A x - b^T x$$

$$\text{st.} \quad Cx = d$$

(opt cond : $\begin{bmatrix} A & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$)

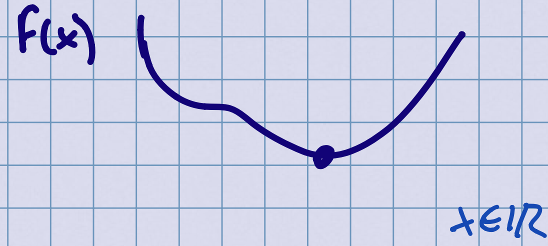
aux variable

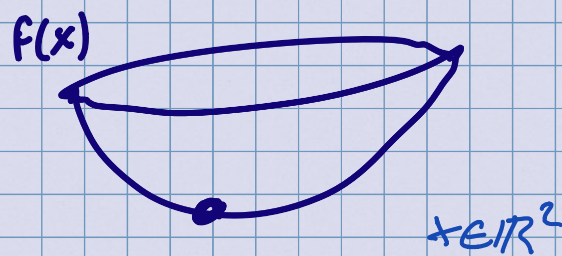
BEYOND QUADRATIC FUNCTIONS

UNCONSTRAINED vs CONSTRAINED

(unconstrained)

$$\min_x \quad f(x)$$





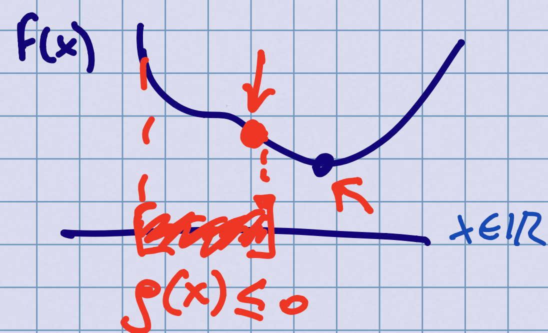
or

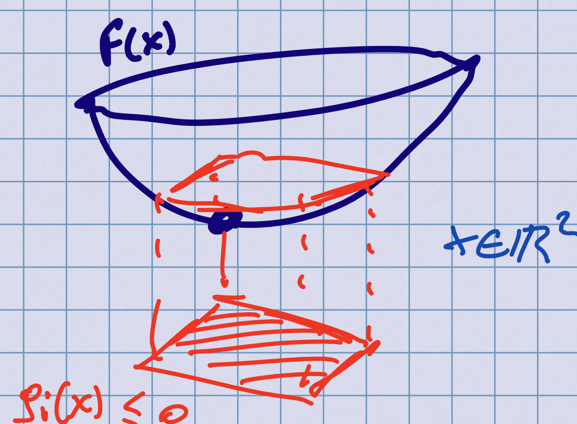
(constrained)

equalities / inequalities

$$\min_x \quad f(x)$$

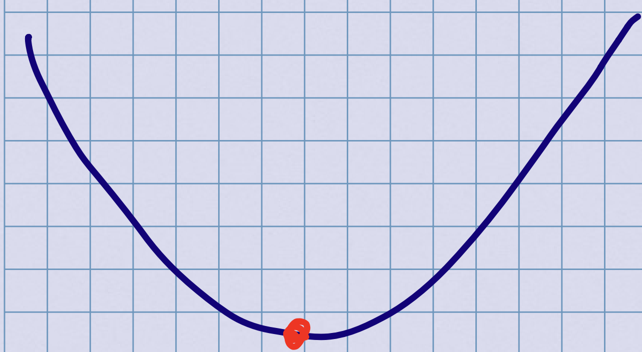
$$\text{s.t.} \quad g_i(x) \leq 0$$





CONVEX vs NON CONVEX

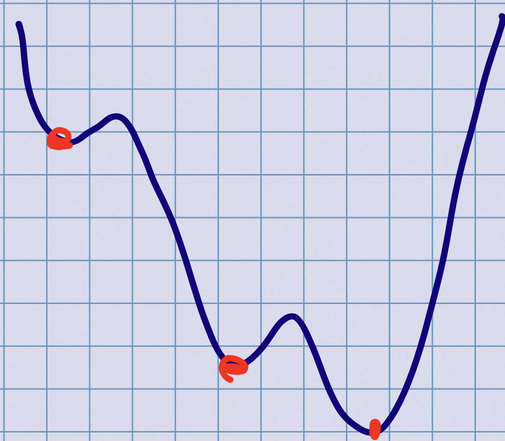
VS



LOCAL MINIMUM $\Rightarrow$ GLOBAL MINIMUM

"EASY" TO (globally) OPTIMIZE

"GOOD" ALGORITHMS, EVEN IN LARGE DIMENSION (POLYNOMIAL TIME)



CAN HAVE MANY LOCAL MINIMIZERS

CAN BE VERY HARD

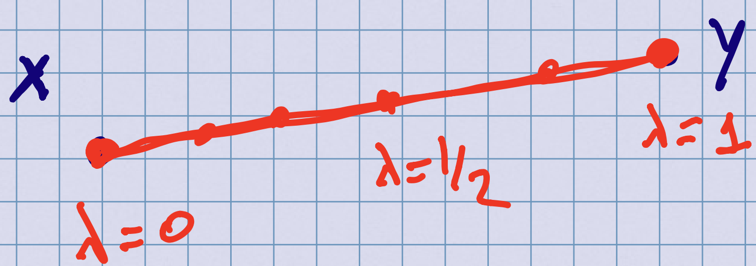
WORST-CASE, ALL KNOWN ALGORITHMS ARE EXPONENTIAL TIME

CONVEX SETS

(in $\mathbb{R}^n$, or any real vector space)

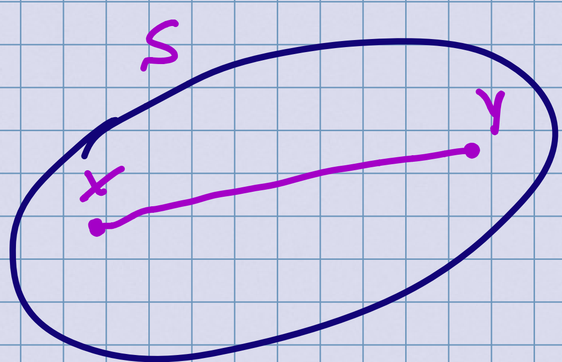
Def: A set S is <u>convex</u> if

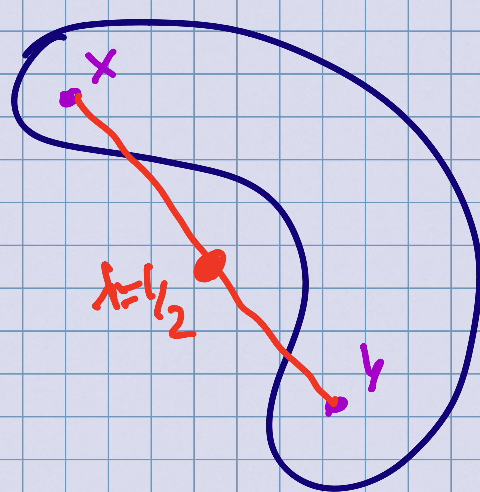
$$x, y \in S \implies (1-\lambda)x + \lambda y \in S \quad \forall \lambda \in [0,1]$$



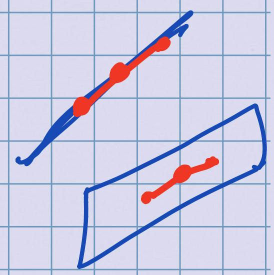
CONVEX

NOT CONVEX





EXAMPLES (of convex sets)

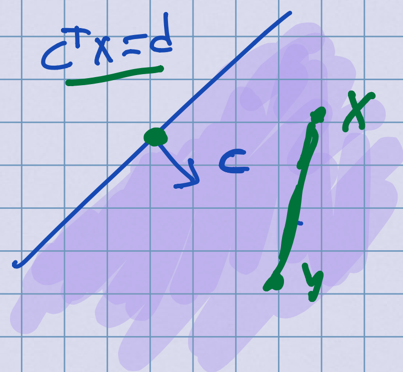


- **SUBSPACES** (affine) $\{x \mid Ax = b\}$

$Ax = b$
$Ay = b$
$\Rightarrow \quad A((1-\lambda)x + \lambda y) = (1-\lambda)\overbrace{Ax}^{b} + \lambda \overbrace{Ay}^{b} = b$

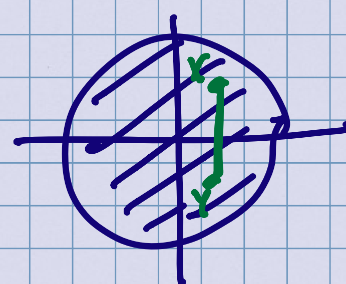
- **HALF-SPACES**

$$c^T x \geq d$$

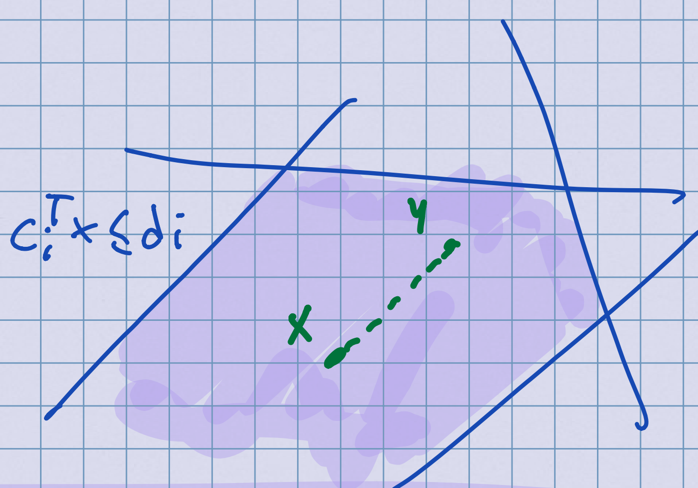


- **BALLS AND ELLIPSOIDS**

$$\|x\|^2 \leq 1 \quad \text{or} \quad \sum x_i^2 \leq 1$$



- **POLYHEDRA**

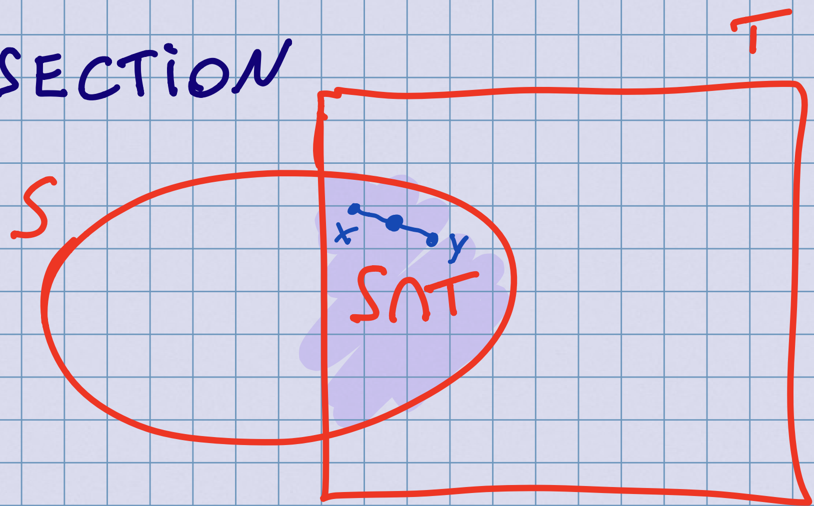


OPERATIONS (between convex sets)

Convex sets are quite nice, because several natural operations preserve convexity:

- INTERSECTION

S convex
T convex

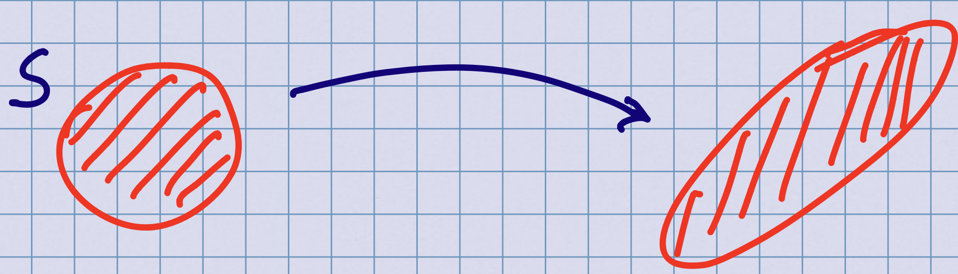


$S \cap T$ is also convex

- LINEAR / AFFINE MAPS

$T: x \rightarrow Ax + b$

$$T(S) = \{ y : y = Ax + b, \ x \in S \}$$



S convex $\Rightarrow T(S)$ convex

$f: \mathbb{R} \to \mathbb{R}$

EXAMPLE: (nonnegative functions)

$$\boxed{f(t) = a + bt^2 + c\sin(t) + d e^t}$$

Q: What is the set S of (a,b,c,d) such that

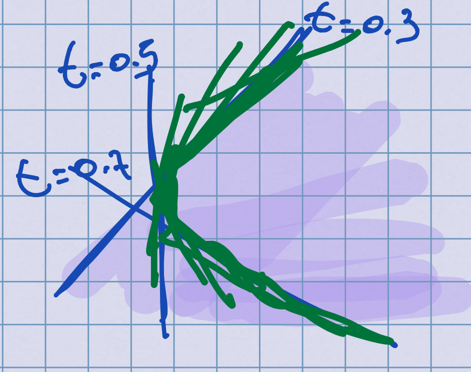
$f(t) \geq 0$ for all $t \in [-1, 1]$?

Claim: S is convex.

$f(t) \geq 0$ for $t \in [-1,1]$

$$\underbrace{a + bt^2 + c\sin(t) + d e^t}_{\text{For fixed } t = t_0 \text{ is a halfspace.}} \geq 0$$

$$p^T \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \geq 0$$



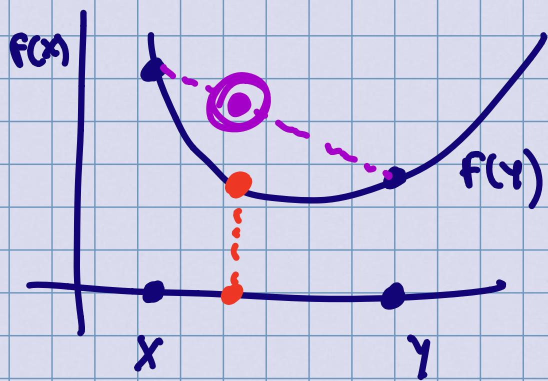
CONVEX FUNCTIONS

$f: \mathbb{R}^n \to \mathbb{R}$

Def: A function f is convex if

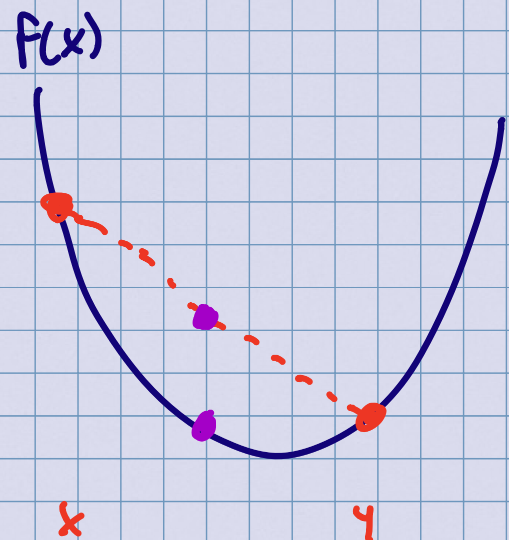
$$f\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)f(x) + \lambda f(y)$$

$\forall x, y \in \mathbb{R}^n, \ \forall \lambda \in [0,1]$

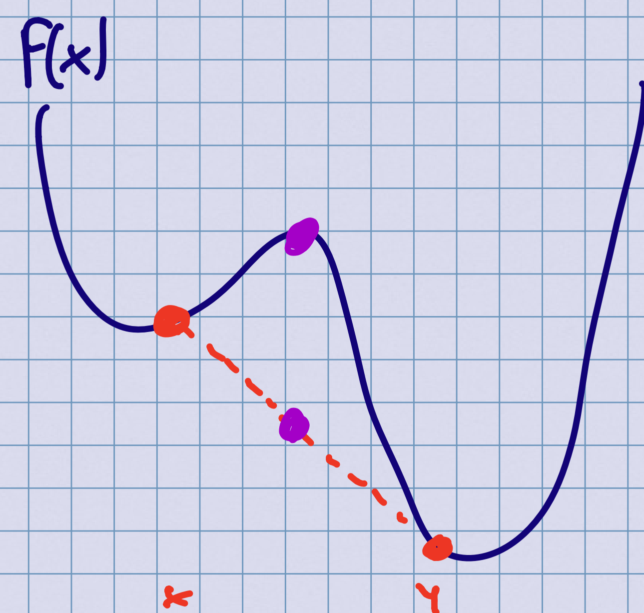


" function is always below the chord "

CONVEX



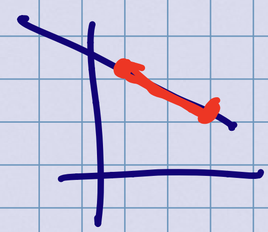
NOT CONVEX



EXAMPLES (of convex functions)

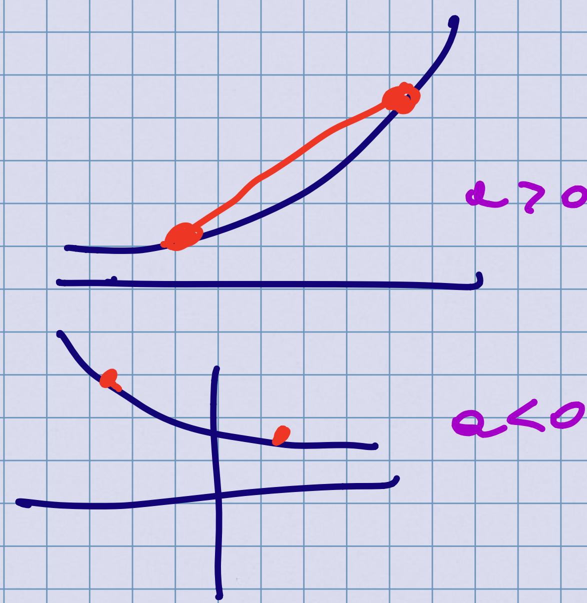
- LINEAR / AFFINE FUNCTIONS

$$f(x) = c^T x + d$$



- EXPONENTIAL

$$f(x) = e^{ax}$$



- QUADRATIC (convex)

$$f(x) = \frac{1}{2} x^T A x + b^T x$$

f(x) is convex <u>if</u> A is <u>psd</u>.

(bad example) $f(x,y) = x^2 - y^2 = \begin{bmatrix} x \\ y \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}}_{A \text{ not psd}} \begin{bmatrix} x \\ y \end{bmatrix}$

OPERATIONS (between convex functions)

Again, certain natural operations preserve convexity!

- (nonnegatively weighted) SUMS

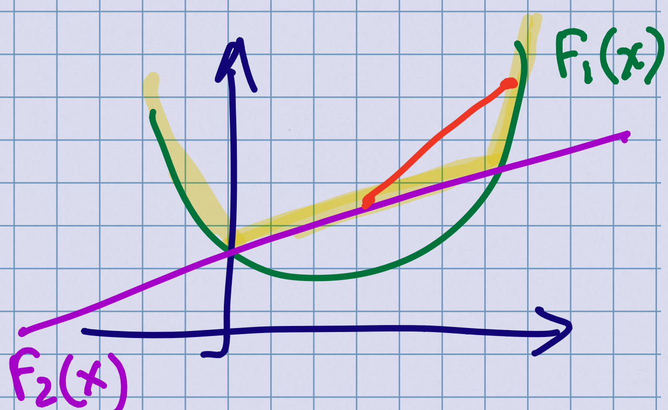
f_1 convex
f_2 convex
$\vdots$
f_n convex

$\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$ convex

$\hookrightarrow$ if $\underline{\alpha_i \geq 0}$.

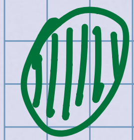
- POINTWISE MAXIMUM

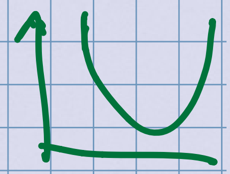
$$g(x) := \max\left(f_1(x), f_2(x)\right)$$



- COMPOSITION w/ AFFINE FUNCTION

$f(x)$ convex $\rightsquigarrow$ $f(Ax+b)$ also convex





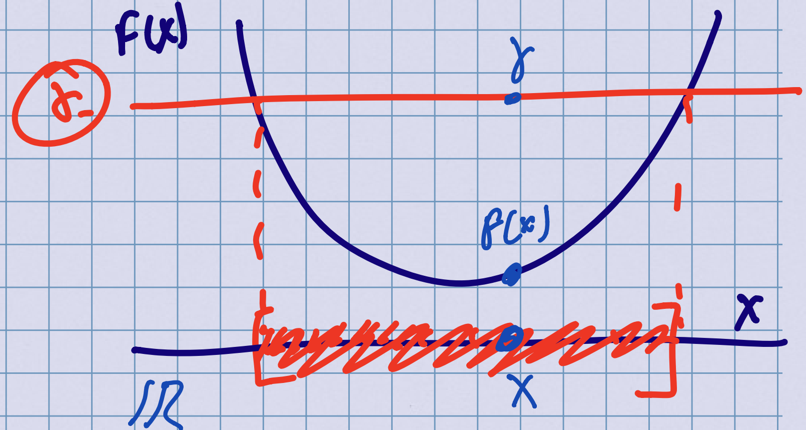
CONVEX SETS $\overset{?}{\Longleftrightarrow}$ CONVEX FUNCTIONS

- **SUBLEVEL SETS** of convex functions are convex sets.

$f : \mathbb{R}^n \to \mathbb{R}$

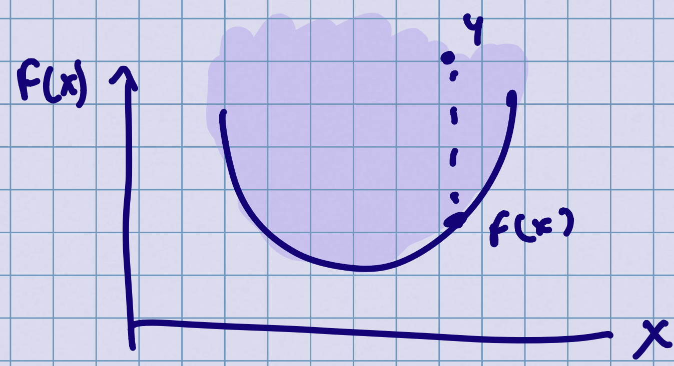
$$S_\gamma = \{ x : f(x) \le \gamma \}$$

$$S_\gamma \subseteq \mathbb{R}^n$$



- **EPIGRAPHS** of convex functions are convex sets.

$$\{ (\overset{\in \mathbb{R}^n}{x}, \overset{\in \mathbb{R}}{y}) : \underbrace{f(x) \le y} \} \subseteq \mathbb{R}^{n+1}$$



CONVEX OPTIMIZATION PROBLEMS

$$\boxed{\begin{aligned} &\min_{x\in\mathbb{R}^n} && f(x) \\ &\text{s.t.} && g_i(x) \leq 0 \qquad i=1\dots m \end{aligned}}$$

where <u>f and g_i</u> are all <u>convex functions</u>.

<u>OBS</u>: The feasible set $\{x\in\mathbb{R}^n : g_i(x)\leq 0\}$ is a <u>convex set</u>.