

Recitation 2

Tuesday September 13, 2022

1 Recap

1.1 Matrix-Vector Form of Linear Equations

We can always translate a system of equations into an equivalent Matrix-vector product. For instance, suppose we have a system of equations

$$\begin{aligned}5x - y - 3z &= 1 \\ -2x + y + 3z &= 4 \\ x - 4y + 6z &= 0\end{aligned}$$

The system is equivalent to

$$\begin{aligned}\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} &= \begin{bmatrix} 5x - y - 3z \\ -2x + y + 3z \\ x - 4y + 6z \end{bmatrix} \\ &= \begin{bmatrix} 5x \\ -2x \\ x \end{bmatrix} + \begin{bmatrix} -y \\ y \\ -4y \end{bmatrix} + \begin{bmatrix} -3z \\ 3z \\ 6z \end{bmatrix} \\ &= x \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} + z \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -1 & -3 \\ -2 & 1 & 3 \\ 1 & -4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.\end{aligned}$$

Therefore, we can rewrite the system as:

$$\begin{bmatrix} 5 & -1 & -3 \\ -2 & 1 & 3 \\ 1 & -4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}.$$

1.2 Solving Linear Equations

As we have seen above, a system of linear equations can be written in matrix-vector form. We can then use **Gaussian elimination** to solve the equations. The matrix we are interested in is the **augmented matrix** $[\mathbf{A} \mid \mathbf{b}]$, where \mathbf{A} is the matrix of linear coefficients, and \mathbf{b} is the vector on the right hand side.

Gaussian elimination is a sequence of elementary row operations performed on the corresponding matrix. Three types of operations are allowed:

1. Swapping two rows,
2. Multiplying a row by a nonzero scalar,
3. Adding a multiple of one row to another row.

In each operation, we want to make the **pivots** go from left to right. The pivot of a row is the leftmost nonzero entry (aka leading coefficient). Eventually, we want the matrix to be in **row echelon form (ref)**. A matrix is in row echelon form if all the pivots go from left to right as we go down the rows. We can also go further to obtain the reduced row echelon form (**Gauss-Jordan elimination**). In the **reduced row echelon form** of a matrix, all the pivots are equal to 1, and every column containing a pivot has zeros elsewhere. If we get a row of zeros after elimination, then the matrix is **singular**, and there might not be a solution.

To solve for general solutions of the system, we can assign each *non-pivot* variable (if any) a **free variable**, and then backsolve for the system. A free variable is a variable that can represent any real numbers.

What if there are rows of zeros in the ref matrix? First of all, those rows have to be at the bottom (why?) In order for solutions to exist, the corresponding entries in **b'** must also be zeros.

1.3 Geometry of linear equations in 3d

$$\underbrace{\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -3 \\ -3 & -1 & 2 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}}_{\mathbf{b}}$$

1.3.1 Row View

Each row poses a linear constraint that describes a plane, so, geometrically, solving the system means finding the intersection of the planes. There are four possibilities:

1. Intersection is a point.
2. Intersection is a line. Two planes intersect in a line, and the third plane contains the line of intersection.
3. Intersection is a plane. All three planes are essentially the same.
4. No intersection. Two or more planes are parallel, or the lines of intersection of each pair are parallel.

1.3.2 Column View

x, y, z describe a linear combination of the column vectors of A that, if possible, gives the vector **b**. There are 3 possibilities:

1. All three columns are linearly independent. **x** is unique.

2. Two columns are linearly independent, and the third column is a linear combination of the other two. The span of the two linearly independent columns is a plane. If \mathbf{b} is on the plane, then there are infinite solutions. Otherwise there is no solution.
3. The columns are scalar multiples of each other. If \mathbf{b} is also a scalar multiple of the column vector, then there are infinite solutions. Otherwise there is no solution.

2 Exercises

1. This exercise is to acquaint us with the idea of "spans".

(a) Translate the following linear system into matrix-vector form.

$$\begin{cases} x + y + z = 1 \\ x + 2y + 3z = 2 \end{cases}$$

- (b) The system has a solution. Fill in the blank. The vector (?) can be written as a linear combination of (?).
- (c) In fact, there is a solution for x, y, z for any vector on the right hand side. Why? Can we turn this observation into a statement about spans?
- (d) Does this new system always have a solution? Why not? What does that mean in terms of the span?

$$\begin{cases} x + y + z = 1 \\ x + 2y + 3z = 2 \\ x + 3y + 5z = 3 \\ x + 4y + 7z = 4 \end{cases}$$

2. For each matrix-vector form of equations $\mathbf{Ax} = \mathbf{b}$ below, identify all pivots. Determine whether it is in row-echelon form. If it is, what are the free variables? Finally, solve the system using the free variables (if any) that you have just denoted.

(a)

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix} \quad \text{with variables } \mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

(b)

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \text{with variables } \mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

(c)

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \text{with variables } \mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

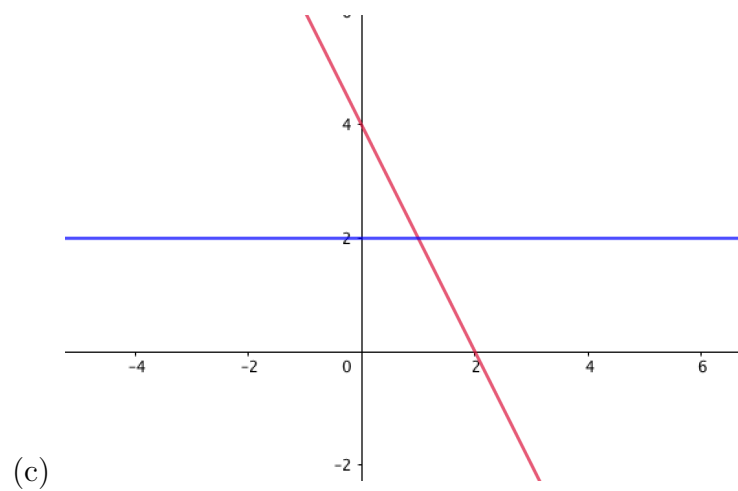
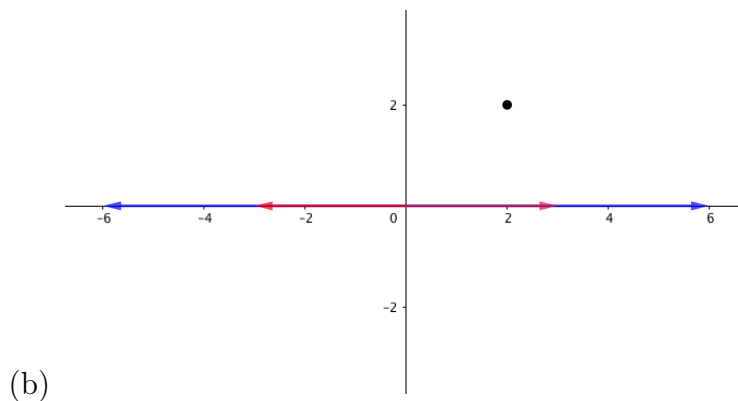
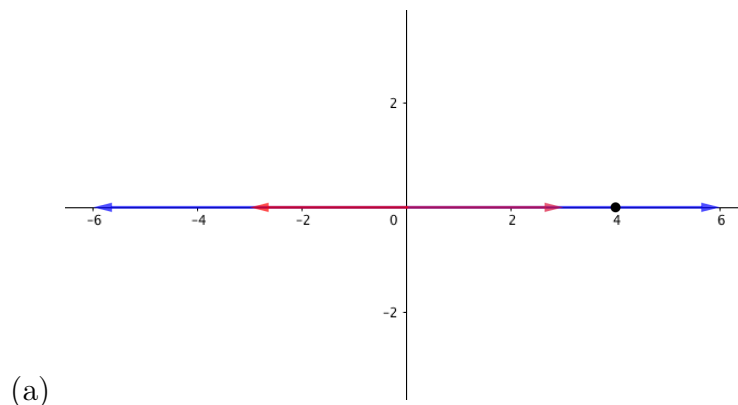
3. Given three following system of equations:

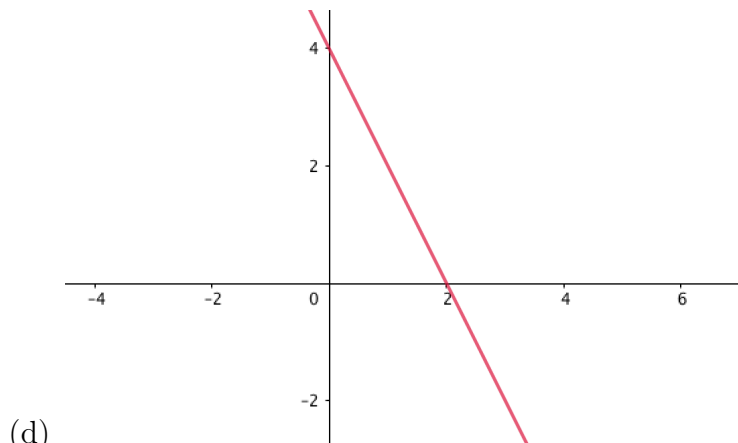
$$\text{System 1: } \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\text{System 2: } \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

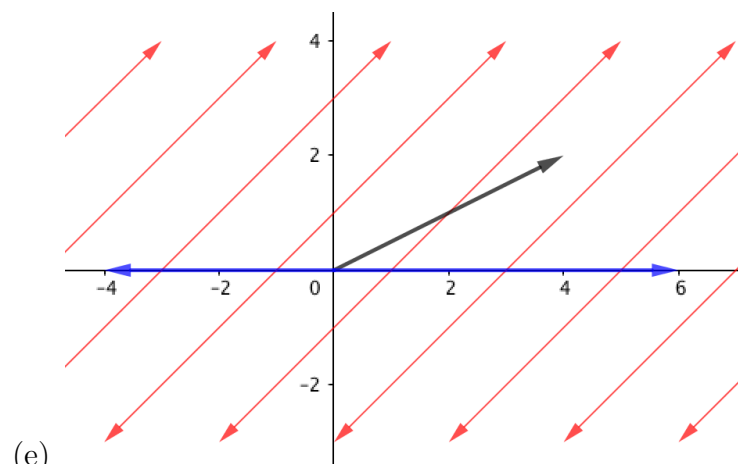
$$\text{System 3: } \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

For each diagram below, determine whether it corresponds to a row/column view of which system.





(d)



(e)

4.

$$x - 2y + 3z = a$$

$$x + y - 3z = b$$

$$3x - 4y + 5z = c$$

- (a) Re-write the equations in matrix-vector form, i.e. $\mathbf{Ax} = \mathbf{b}$. Is \mathbf{A} singular?
- (b) Solve for x, y, z when $a = 1, b = 7, c = 8$
- (c) Solve for x, y, z when $a = 4, b = -2, c = 8$
- (d) Solve for x, y, z when $a = 1, b = 7, c = 7$

5.

$$\mathbf{A} = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array} \right]$$

- (a) After two row operations on \mathbf{A} , we end up with $\mathbf{B} = \left[\begin{array}{ccc|c} -3 & -1 & 2 & -11 \\ 2 & 1 & -1 & 8 \\ 0 & 2 & 1 & 5 \end{array} \right]$.

What are the two row operations?

- (b) Convert \mathbf{A} to row echelon form.
- (c) Re-write the row echelon form of \mathbf{A} to linear equations and solve.

6.

$$x + 3y - 2z = 5$$

$$3x + 5y + 6z = 7$$

$$2x + 4y + 3z = 8$$

- (a) What is the augmented matrix $[A \mid b]$ of the system?
- (b) Find the reduced row echelon form of the augmented matrix. Is \mathbf{A} singular?
- (c) Solve for x, y, z .

3 Solutions

1. (a)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(b) the vector $(1,2)$ can be written as a linear combination of $(1,1)$, $(1,2)$, and $(1,3)$.

(c) $(1,1)$, $(1,2)$, and $(1,3)$ span R^2 .

(d) No the system does not always have a solution, because $(1,1,1,1)$, $(1,2,3,4)$, $(1,3,5,7)$ does not span R^4 .

2. (a) Pivots are 1 (row 1 column 1), 2 (row 2 column 2), and 3 (row 3 column 3). \mathbf{A} is in a row echelon form. There is no free variable.

The last equation is $3c = -3$ which gives $c = -1$. The second equation is $2b + c = 3$. Plugging in $c = -1$, we have $b = 1$. Finally, the first equation in $a + 3b + c = 2$. Plugging in $b = 1$ and $c = -1$, we will have $a = 0$.

(b) Pivots are 1 (row 1 column 1), 2 (row 2 column 2), and 1 (row 3 column 2). \mathbf{A} is *not* in a row echelon form.

(c) Pivots are 1 (row 1 column 1), 2 (row 2 column 2), and 1 (row 3 column 3). \mathbf{A} is in a row echelon form. Pivots are in columns 1, 2, and 3, which corresponds to variables a, b , and c . Therefore, the only free variable is d .

The last equation yields $c = 1$. The second equation is $2b + c - d = -1$ which gives $b = \frac{d-2}{2}$. The first equation is $a + 4b + d = 0$ which gives $a = -4b - d = -4\left(\frac{d-2}{2}\right) - d = -3d + 4$.

3. (a) System 3, column view.

(b) System 2, column view.

(c) System 1, row view.

(d) System 3, row view.

(e) System 1, column view.

4. (a)

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & -3 \\ 3 & -4 & 5 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\mathbf{b}}$$

(b) Row echelon form (one possible solution): $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$. Since the last row only consists of 0s, \mathbf{A} is singular.

(c) No solution.

(d) $x = 1, y = 0, z = 1$. Note that $\begin{bmatrix} 4 \\ -2 \\ 8 \end{bmatrix}$ is the sum of column 1 and column 3.

(e) Reduced row echelon form:
$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

z is a free variable, meaning it can be any real number. $x = z + 5, y = 2z + 2$.

5. (a) First, exchange row 1 and row 2. Second, add row 2 to row 3.

(b) A possible solution is
$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

6.

$$2x + y - z = 8$$

$$y + z = 2$$

$$z = -1$$

$$x = 2, y = 3, z = -1$$

(a)
$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array} \right]$$

(b)
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array} \right]. \mathbf{A} \text{ is not singular.}$$

(c) $x = -15, y = 8, z = 2$