Recitation 3

Thursday September 14, 2022

1 Recap

1.1 Reduced Row Echelon Form

Reduced row echelon form (rref) is a special case of the row echelon form (ref). A matrix is in a reduced row echelon form if and only if it satisfies the following three conditions.

1. It is in a row echelon form (ref).
2. All the pivots are equal to 1.
3. Every column containing a pivot has zeros elsewhere.

Once we have an ref matrix, we can perform additional steps of row operations to obtain an rref matrix.

1.2 Matrix Multiplication

How to do matrix multiplication? In other words, if we are given matrices A and B, how do we determine another matrix C such that $AB = C$? First of all, we note that the inner-dimensions of A and B must match. Let's say A is an $m \times n$ matrix, and B is an $n \times p$ matrix. This results in C being an $m \times p$ matrix.

There are multiple ways to interpret matrix multiplication.

1. **Entry-wise:** For each $1 \leq i \leq m$ and $1 \leq k \leq p$, we have
$$C_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk}.$$
2. **Inner product:** C_{ik} is the inner product of the i^{th} row in A and the k^{th} column in B.
3. **Column-wise:** the i^{th} column of matrix C is a matrix-vector product of A and the i^{th} column of B. In other words, if $B = \begin{bmatrix} B_1 & B_2 & \dots & B_p \end{bmatrix}$ where B_i is the i^{th} column of B, then
$$C = \begin{bmatrix} AB_1 & AB_2 & \dots & AB_p \end{bmatrix}.$$

4. **Outer product:** C is the sum of the product of i^{th} column of A and the i^{th} row of B – ranging from $i = 1$ to n. In other words, let's say

$$A = \begin{bmatrix} A_1 & A_2 & \dots & A_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \dots \\ B_n \end{bmatrix}$$

where A_i is the i^{th} *column* of A and B_j is the j^{th} row of B. Then,

$$C = \sum_{i=1}^{n} A_i B_i.$$

Each $A_i B_i$ is an $m \times p$ matrix itself.

1.3 Properties of Matrix Multiplication

1. Associative: $A(BC) = (AB)C$.
2. Distributive:
   - $A(B + C) = AB + AC$
   - $(A + B)C = AC + BC$
3. Non-commutative (in general): $AB \neq BA$.
4. Identity Matrix: I_n is an $n \times n$ square matrix with 1's on the diagonal and 0's elsewhere. For any $m \times n$ matrix A, $I_m A = A$; $AI_n = A$.

2 Exercises

1. For each of the following row echelon form (ref) augmented matrix $\begin{bmatrix} \mathbf{A} \mid \mathbf{b} \end{bmatrix}$, perform row operations to obtain a reduced row echelon form (rref) one. Use the rref matrix to solve for general solutions $\mathbf{x}$. You can use any appropriate free variables, if needed.

   (a) $\begin{bmatrix} \mathbf{A} \mid \mathbf{b} \end{bmatrix} = \left[\begin{array}{ccc|c} 2 & 5 & -2 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 5 & 10 \end{array}\right]$ with variables $\mathbf{x} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$.

   (b) $\begin{bmatrix} \mathbf{A} \mid \mathbf{b} \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 1 \\ 0 & 0 & -2 & 1 & 5 \\ 0 & 0 & 0 & 0 & 99 \end{array}\right]$ with variables $\mathbf{x} = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$.

   (c) $\begin{bmatrix} \mathbf{A} \mid \mathbf{b} \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 1 \\ 0 & 0 & -2 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ with variables $\mathbf{x} = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$.

(d) $\left[\mathbf{A} \mid \mathbf{b}\right] = \left[\begin{array}{cccc|c} -2 & 1 & 3 & 0 & -3 \\ 0 & 2 & -7 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \end{array}\right]$ with variables $\mathbf{x} = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$.

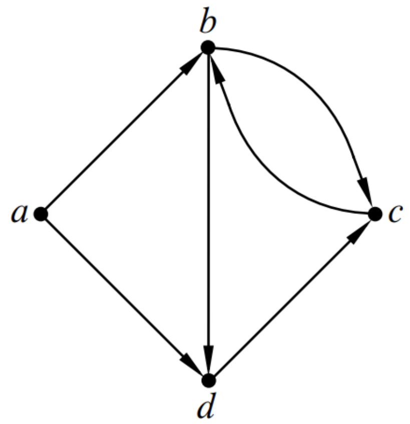
2. Let A and B be arbitrary $n \times n$ matrices. Indiate whether the following statements are True or False.

   (a) $AB = BA$.

   (b) If AB is a zero matrix (aka every entry is 0), then either A or B is a zero matrix.

   (c) If $AB = BA$, then either $A = I_n$ or $B = I_n$.

3. Counting Walks
   In this problem we will explore directed graphs where each edge points from one vertex, the “head", to another, the “tail". Now the only allowable walks will be ones that traverse the edges in the forwards direction. You can think of them as one-way streets.



   (a) Write the adjacency matrix A for the graph. Note the edge from a to b is not the same as the edge from b to a.

   (b) Are there walks of length 2 that start at node a and end at node b? If so, how many? What about from node a to c?

   (c) Are there walks of length 3 that start at node a and end at b? If so, how many?

   (d) How do you interpret $A + I_4$ from an aspect of the graph?

   Hint: Adding I_4 is equivalent to adding 4 lines to the graph. But which lines are they?

   (e) In class, we established that entries of A^2 represent the number of length-2 walks. What do the entries of $(A+I)^2$ represent? What can we tell if an entry is zero/non-zero?

   (f) Suppose we have a gigantic graph G and we want to check if there exists any walk of length *at most* l that goes from node u to v. How do we do that?