Linear Algebra and Optimization
MIT Fall 2022

Recitation 5

Thursday September 22, 2022

1 Recap

1.1 Vector Space

A vector space V is a set that is closed under **vector addition** and **scalar multiplication**. These two operations must satisfy a set of axioms[1]. A basic example is the real vector space $\mathbb{R}^n$, where

- every vector is represented by a list of n real numbers
- scalars are real numbers
- addition is defined element-wise
- scalar multiplication is multiplication on each term separately

For a general vector space, the scalars are elements of a field F (e.g., the complex numbers), in which case V is called a vector space over F. If $W \subseteq V$ is also a vector space with respect to the operations in V, then W is called a subspace of V.

Key Fact. If S_1 and S_2 both are vector space, then $S = S_1 \cap S_2$ is a subspace.

1.2 Column Space

The **column space** of an $m \times n$ matrix A, denoted $C(A)$, is the set of all **linear combinations** of columns of A, or the **span** of A.

- $C(A) = \{Ax \mid x \in \mathbb{R}^n\}$.
- $Ax = b$ has a solution iff $b \in C(A)$.
- If $m = n$, then A is invertible iff $C(A) = \mathbb{R}^n$.

1.3 Null Space

The **null space** of A, denoted $N(A)$, is the set of vectors x such that $Ax = 0$.

- $N(A) = \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\}$.
- If B is a square and invertible matrix, then $N(A) = N(BA)$.
- If A is $n \times n$, then $C(A) = \mathbb{R}^n \iff N(A) = \{\mathbf{0}\} \iff A$ is invertible.

[1] https://en.wikipedia.org/wiki/Vector$_{s}paceDefinition, ormoreformallyathttps : //encyclopediaofmath.org/wiki/V$

2 Exercises

1. Is V a real vector space?
   (a) V is the set of all n-dimensional vectors with positive entries, with usual vector operations.
   (b) V is the set of all n-dimensional vectors whose elements sum to 0, with usual vector operations.
   (c) V is the set of all $n \times n$ diagonal matrices, with usual matrix operations.
   (d) V is the set of all polynomials with degree up to d, with usual polynomial operations.
   (e) V is the set of all constant functions, i.e. $f(x) = c$ for some constant $c \in \mathbb{R}$, with usual real number operations.
   (f) V is the set of all single-variable polynomial whose value at 0 is 1, i.e. polynomial P with $P(0) = 1$, with usual polynomial operations.
2. True or False
   (a) Define the row space of matrix A as the span of the row vectors of A. If A is a square matrix, then the row space of A equals the column space.
   (b) The row space of A is equal to the column space of A^T.
   (c) If the row space of A equals the column space, then $A^T = A$.
   (d) A 4 by 4 permutation matrix has column space equal to $\mathbb{R}^4$.
   (e) Let $v \in N(A)$. If x is a solution to equation $A\mathbf{x} = b$, so is $x + v$.
3. A is a 3 by 3 matrix.
   (a) $x = [1\ 2\ 3]^T$ is in the null space of some matrix A. What is $A(2x)$?
   (b) $y = [1\ 2\ 4]^T$ is also in the null space of A. What is $A(2x + y)$?
   (c) $Az = [1\ 2\ 5]^T$. What is $A(2x + y + z)$
4. $Ax = b$, A is a 3 by 3 matrix, x and b are 3-dimensional vectors.
   (a) When $b = [1\ 2\ 3]^T$ we have infinite solutions. Is b in the column space of A?
   (b) Assume (a) is still true. Is A invertible?
   (c) Now assume A is invertible, do we have a solution when $b = \mathbf{0}$? How many?
5.
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$
   (a) Find a nonzero vector in the null space of A, if any exists.
   (b) Is A invertible?
   (c) Find a vector in the column space of A that is not equal to either column.
   (d) Find a vector in the row space of A that is not equal to either row.