Recitation 4

Tuesday September 20, 2022

1 Recap

1.1 Matrix Inverses

1. Existence: for some $m \times n$ matrix A,

   - if $m < n$, A may have a right inverse B such that $AB = I_m$;
   - if $m > n$, A may have a left inverse C such that $CA = I_n$; *Note that both inverses have dimensions* $n \times m$
   - if $m = n$, A may have both a right and left inverse (not guaranteed). Then the left and right inverses are equal, and we say that A is invertible.

     In other words, if A is a $n \times n$ matrix and is invertible, then the following statements are equivalent: 1) $B = A^{-1}$, 2) $AB = I_n$, and 3) $BA = I_n$.

2. How to find the inverse of a square matrix A using Gauss-Jordan elimination:

   (a) If we get a row of zeros in the reduced row echelon form (RREF), then the matrix is singular and does not have an inverse. Otherwise the RREF form of an invertible square matrix is the identity matrix I_n.

   (b) Since the row operations are linear with respect to each row, we can describe them using an $n \times n$ matrix E such that $EA = I_n$. Right-multiply the equation by A^{-1} yields $E = A^{-1}$. We can find the inverse matrix by performing the same set of row operations in the Gauss-Jordan elimination process to I_n.

1.2 Linear Map

We say $f : \mathbb{R}^n \to \mathbb{R}^n$ is a *linear map* if and only if there exists an $n \times n$ matrix A such that $f(x) = Ax$ for any n-dimensional vector x.

In the upcoming sections, we will see some examples of linear maps. They include permutation, rotation, and projection operations where the matrix A corresponds to permutation, rotation, and projection matrix respectively.

1.3 Fundamental Operations

1. **Length and Angle**

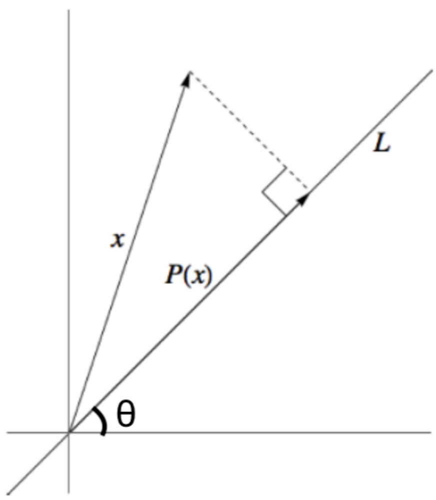
   - The length of an n-dimensional vector v: $\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$.
   - The angle θ between two vectors u and v: $\cos(\theta) = \frac{\langle u,v \rangle}{\|u\| \cdot \|v\|}$

2. **Permutation**
   A permutation matrix is a square matrix with 0's and 1's, where there is exactly one entry equal to 1 in each row and column. In a matrix-vector product, the permutation matrix rearranges the entries of the vector:
$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ x \\ y \end{bmatrix}$$
   In particular, if we want to map the i^{th} entry of an original vector to the $\pi(i)^{\text{th}}$ entry of a new vector, we can write its corresponding permutation matrix A with entries
$$A_{i,j} = \mathbb{1}_{\pi(j)=i}$$

3. **Transpose**
   The transpose of an $m \times n$ matrix A is an $n \times m$ matrix A^T such that $(A^T)_{i,j} = A_{j,i}$. In other words, the ith row of A is the ith column of A^T, and vice versa. There are some special properties of the transpose:
   - $(AB)^T = B^T A^T$
   - $(A^T)^T = A$
   - $(A + B)^T = A^T + B^T$
   - If A is a permutation matrix, then $A^T = A^{-1}$.

4. **Rotation**
   To rotate a 2d vector counter-clockwise by some angle θ, we apply the rotation matrix
$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
   In addition, the inverse of R_θ is $R_{-\theta} = R_\theta^T$.

5. **Projection**
   A projection matrix maps a vector x to its projection onto a line L.



In lecture, we talked about $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ mapping x to the x-axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$$

Then, for some line L with angle θ from the x-axis, the projection matrix is

$$P_\theta = R_\theta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R_{-\theta} = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}.$$

The projection matrix, however, does not preserve length nor is it an invertible transformation.

2 Exercises

1. Find the angle between vectors $u = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 4 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 2 \end{bmatrix}$.

2. A is a permutation matrix such that $A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \end{bmatrix}$. Find A and A^{-1}.

3. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Find the left and right inverses of A and B, or show that they don't exist.

4. In this exercise, we will derive a reflection matrix (Ref) which given a vector v, its reflection across a line L with angle θ from x-axis is $\mathrm{Ref}_\theta \cdot v$. To do so, answer the following subparts in order.

   (a) What is a reflection of $v = \begin{bmatrix} x \\ y \end{bmatrix}$ across x-axis?

   (b) What is a reflection matrix Ref_0 that reflects any 2-dimensional vector v across x-axis? In other words, determine a 2×2 matrix Ref_0 such that a reflection of 2-dimensional vector v across x-axis is $\mathrm{Ref}_0 \cdot v$.

   (c) Now we have a line L passing through the origin with angle θ with x-axis. Determine a reflection matrix Ref_θ that reflects any 2-dimensional vector v across L?

   Hint: Try to mimic the derivation of projection matrix P_θ.

   (d) Explain (both algebraically and geometrically) why $\mathrm{Ref}_\theta = 2P_\theta - I$.

(e) Does the reflection matrix preserves length? Is it invertible? Try to explain the answer both algebraically and geometrically.

5. Let both A and B be $n \times n$ matrices. Determine whether the following statements are True of False.

   (a) If A is invertible, then we can solve $AX = B$ by $X = BA^{-1}$.

   (b) If both A and B are invertible, then AB is invertible.

   (c) If AB is invertible, then both A and B are invertible.

   (d) If AB is invertible, then $(AB)^{-1} = A^{-1}B^{-1}$.

   (e) If both A and B are permutation matrices, then AB is also a permutation matrix.

   (f) If P is a projection matrix, then $P^2 = P$.

3 Solutions

1. $\|u\| = \sqrt{0^2+3^2+1^2+4^2} = \sqrt{26}$, $\|v\| = \sqrt{2^2+2^2+(-1)^2+2^2} = \sqrt{13}$, and $\langle u, v\rangle = 0\cdot 2 + 3\cdot 2 + 1\cdot(-1) + 4\cdot 2 = 13$. So we have $\cos(\theta) = \frac{13}{\sqrt{26}\cdot\sqrt{13}} = \frac{1}{\sqrt{2}}$ which implies $\theta = \pi/4$.

2. The permutation maps the 1^{st} to 4^{th} entry, 2^{nd} to 2^{nd} entry, 3^{rd} to 1^{st} entry, and 4^{th} to 3^{rd} entry. We can write this permutation as $\pi(1) = 4, \pi(2) = 2, \pi(3) = 1$, and $\pi(4) = 3$. This means $A_{4,1} = A_{2,2} = A_{1,3} = A_{3,4} = 1$ while the remaining entries of A are all 0. So $A = \begin{bmatrix} 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1\\ 1&0&0&0 \end{bmatrix}$, and then $A^{-1} = A^T = \begin{bmatrix} 0&0&0&1\\ 0&1&0&0\\ 1&0&0&0\\ 0&0&1&0 \end{bmatrix}$.

3. A is a square matrix. If A is invertible, then it has both a left and right inverse, and the two inverses are equal.

   To derive A^{-1}, we perform row operations on A until we get I_3. Then we perform the same sequence of operations on I_3 to get A^{-1}.

   The same procedure also applies for B with appropriate dimension I_5.

   (a) $$A = \begin{bmatrix} 1&2&3\\ 0&1&0\\ 0&0&1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix} 1&0&3\\ 0&1&0\\ 0&0&1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 3R_3} \begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&0&1 \end{bmatrix} = I_3.$$
   $$I_3 = \begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&0&1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix} 1&-2&0\\ 0&1&0\\ 0&0&1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 3R_3} \begin{bmatrix} 1&-2&-3\\ 0&1&0\\ 0&0&1 \end{bmatrix} = A^{-1}.$$

   (b) $$B = \begin{bmatrix} 1&0&0&0&0\\ 0&2&0&0&0\\ 0&0&3&0&0\\ 0&0&0&4&0\\ 0&0&0&0&5 \end{bmatrix} \xrightarrow{\substack{R_2 \leftarrow R_2/2\\ R_3 \leftarrow R_3/3\\ R_4 \leftarrow R_4/4\\ R_5 \leftarrow R_5/5}} \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix} = I_5.$$
   $$I_5 = \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix} \xrightarrow{\substack{R_2 \leftarrow R_2/2\\ R_3 \leftarrow R_3/3\\ R_4 \leftarrow R_4/4\\ R_5 \leftarrow R_5/5}} \begin{bmatrix} 1&0&0&0&0\\ 0&1/2&0&0&0\\ 0&0&1/3&0&0\\ 0&0&0&1/4&0\\ 0&0&0&0&1/5 \end{bmatrix} = B^{-1}.$$

4. (a) A reflection across x-axis preserves the value of x-coordinate, but flips the sign of y-coordinate. This means $v = \begin{bmatrix} x\\ y \end{bmatrix}$ is reflected into $\begin{bmatrix} x\\ -y \end{bmatrix}$.

   (b) As $v = \begin{bmatrix} x\\ y \end{bmatrix}$ gets reflected across x-axis into $\begin{bmatrix} x\\ -y \end{bmatrix}$, we want to ask which 2×2 matrix Ref_0 gives $\text{Ref}_0 \cdot \begin{bmatrix} x\\ y \end{bmatrix} = \begin{bmatrix} x\\ -y \end{bmatrix}$. The equation implies $\text{Ref}_0 = \begin{bmatrix} 1&0\\ 0&-1 \end{bmatrix}$.

   (c) Similar to projection, we can derive a reflection matrix for arbitrary angle θ by 1) rotate clockwise with angle θ, 2) reflect across x-axis, and 3) rotate counterclockwise with angle θ. This gives

$$\text{Ref}_\theta = R_\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_{-\theta}.$$

(d) Algebraically,

$$\begin{aligned}
\text{Ref}_\theta + I &= R_\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_{-\theta} + I \\
&= R_\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_{-\theta} + R_\theta I R_{-\theta} \\
&= R_\theta \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + I \right) R_{-\theta} \\
&= R_\theta \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} R_{-\theta} \\
&= 2 \cdot R_\theta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R_{-\theta} \\
&= 2 \cdot P_\theta.
\end{aligned}$$

This is equivalent to $\text{Ref}_\theta = 2P_\theta - I$.

Geometrically, we can interpret the projection of v on line L to be the midpoint between v and v's reflection across L. We also know that the projection is $P_\theta \cdot v$ and the reflection is $\text{Ref}_\theta \cdot v$. Thus, we can write

$$P_\theta \cdot v = \frac{1}{2} (\text{Ref}_\theta \cdot v + v) = \left(\frac{\text{Ref}_\theta + I}{2} \right) v.$$

Since the equation holds for arbitrary v, we then have $P_\theta = \frac{\text{Ref}_\theta + I}{2}$ which is equivalent to $\text{Ref}_\theta = 2P_\theta - I$.

(e) The reflection matrix does preserve the length and is invertible.

Algebraically, matrices R_θ, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $R_{-\theta}$ have both properties so their product also does.

Geometrically, the reflection of a vector across a line maintains the length (think of reflecting as looking into a mirror – an object has the same length as it is in the mirror.)

Given any reflection point, we can uniquely determine its original point (by reflecting it back). This means the reflection matrix is invertible.

5. (a) False. To solve for X, we have to *left* multiply $AX = B$ by A^{-1} which gives $X = A^{-1}B$.

(b) True. Consider $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$. This means AB has a *right* inverse which is $B^{-1}A^{-1}$. But since AB is also a square matrix, we can say that AB is invertible.

(c) True. Since AB is invertible, it must have an inverse – say a $n \times n$ matrix C that $(AB)C = C(AB) = I$. We can rearrange it into $A(BC) = I$ and $(CA)B = I$ which means A has a right inverse and B has a left inverse. Since both of them are square matrices, they both are invertible.

(d) False. Part a) implies that $(AB)^{-1} = B^{-1}A^{-1}$ which is defined since both B^{-1} and A^{-1} are defined.

(e) True. Let's say we have a n-dimensional vector v. Then $(AB)v = A(Bv)$ which is equivalent to v undergoing a permutation π_B of B, and then undergoing another permutation π_A of A. This means AB is a permutation matrix of the permutation $\pi_A \circ \pi_B$.

(f) True. Let v be an arbitrary vector. First we note that Pv is a projection of v onto L – which is already on the line L. Applying $P^2v = P(Pv)$ is equivalent to projecting Pv (which is already on L) onto L. The projection is thus Pv. This gives $P^2v = Pv$ for arbitrary v which means $P^2 = P$.