# Recitation 6

*Tuesday September 27, 2022*

## 1 Recap

### 1.1 Linear Independence

A collection of vectors $\{v_1, \dots, v_n\}$ is *linearly independent (LI)* if any linear combination that results in a zero vector must be trivial. In other words,

$$\sum_{i=1}^{n} \lambda_i v_i = \mathbf{0} \quad \Longrightarrow \quad \lambda_i = 0 \text{ for all } i = 1, \dots, n.$$

We also say that a collection of vectors $\{v_1, ..., v_n\}$ is *linearly dependent* if it is not LI. In other words, there exists scalar multipliers $\lambda_1, \dots, \lambda_n$ where at least one of them is non-zero, such that

$$\sum_{i=1}^{n} \lambda_i v_i = \mathbf{0}.$$

**Key Fact:** There can be at most $n$ linearly independent vectors in $\mathbb{R}^n$.

### 1.2 Generators

Let $\mathcal{S}$ be a subspace. We say that $\{v_1, \dots, v_k\} \subset \mathcal{S}$ are *generators* of $\mathcal{S}$ if every vector $v \in \mathcal{S}$ is a linear combination of $\{v_1, \dots, v_k\}$. In other words, $v = \lambda_1 v_1 + \cdots + \lambda_k v_k$ for some scalars $\lambda_1, \dots, \lambda_k$.
We can also write $\mathcal{S} = \langle v_1, \dots, v_k \rangle$ or $\mathcal{S} = \mathrm{Span}(v_1, \dots, v_k)$

### 1.3 Two Descriptions of Subspaces

Two useful descriptions of subspaces include:

1. Equations. We can describe a subspace $\mathcal{S}$ as a set of vectors satisfying certain linear relationships between their entries.

2. Generators. We can describe a subspace $\mathcal{S}$ as the span of a set of vectors.

For instance, the subspace $\mathcal{S}$ of 3-dimensional vectors whose third entry is the sum of the first and second entries can be expressed as either:

$$\mathcal{S} = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \;:\; v_1 + v_2 - v_3 = 0 \right\} \qquad \text{or} \qquad \mathcal{S} = \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Depending on the task, one description may be more convenient than the other. We can use Gaussian elimination to go from one description to the other.

### 1.4 Basis

We say $\{v_1, \ldots, v_k\}$ is a *basis* of a subspace $\mathcal{S}$ if they generate $\mathcal{S}$ and are LI.
In general, a subspace has infinitely many different bases. However, they all must have the same cardinality (i.e., the number of vectors in the basis) – this is called the *dimension* of the subspace.

## 2 Exercises

1. Note that the notions of linear independence and linear dependence are not quite symmetric. In particular:

   (a) Show that if $\{v_1, \ldots, v_k\}$ are LI, then any subset of the vectors is also LI.

   (b) Does a similar statement hold for linear dependence? Prove this, or give a counterexample.

2. Identify if the following sets of vectors are linearly independent or not.

   (a) $A = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \end{bmatrix} \right\}$.

   (b) $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

   (c) $C = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$

   (d) $D = \left\{ \begin{bmatrix} 0 \\ -1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 11 \\ -6 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -12 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 1000 \\ 100 \\ 10 \\ 1 \end{bmatrix} \right\}$.

3. Let $A = \begin{bmatrix} -3 & 1 & 0 & 5 \\ -2 & 2 & -2 & 1 \\ 1 & -3 & 4 & 3 \end{bmatrix}$. Answer the following questions.

   (a) Are the columns of $A$ linearly independent?

   (b) Find a set of generators for $N(A)$, the nullspace of $A$.

   (c) Find a basis of the column space $C(A)$, aka the span of columns.

   You can do it either by inspection, or algorithmically, using Gauss-Jordan elimination.

4. Consider the two subspaces

$$U = \text{Span}\left\{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\3\\1\\2\end{bmatrix}\right\}$$

$$V = \left\{\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} : v_1 + 2v_2 - 4v_3 = 0\right\}$$

(a) Write a description of $U$ as a set of vectors that satisfy linear relationships.

(b) Write a description of $V$ as the span of a set of generators.

(c) Compute the dimension and a basis of $U$ and $V$.

(d) Compute the dimension and a basis of $U \cap V$.