Recitation 6

Tuesday September 27, 2022

1 Recap

1.1 Linear Independence

A collection of vectors $\{v_1, \ldots, v_n\}$ is *linearly independent (LI)* if any linear combination that results in a zero vector must be trivial. In other words,

$$\sum_{i=1}^{n} \lambda_i v_i = \mathbf{0} \quad \Longrightarrow \quad \lambda_i = 0 \text{ for all } i = 1, \ldots, n.$$

We also say that a collection of vectors $\{v_1, ..., v_n\}$ is *linearly dependent* if it is not LI. In other words, there exists scalar multipliers $\lambda_1, \ldots, \lambda_n$ where at least one of them is non-zero, such that

$$\sum_{i=1}^{n} \lambda_i v_i = \mathbf{0}.$$

Key Fact: There can be at most n linearly independent vectors in $\mathbb{R}^n$.

1.2 Generators

Let $\mathcal{S}$ be a subspace. We say that $\{v_1, \ldots, v_k\} \subset \mathcal{S}$ are *generators* of $\mathcal{S}$ if every vector $v \in \mathcal{S}$ is a linear combination of $\{v_1, \ldots, v_k\}$. In other words, $v = \lambda_1 v_1 + \cdots + \lambda_k v_k$ for some scalars $\lambda_1, \ldots, \lambda_k$.
We can also write $\mathcal{S} = \langle v_1, \ldots, v_k \rangle$ or $\mathcal{S} = \mathrm{Span}(v_1, \ldots, v_k)$

1.3 Two Descriptions of Subspaces

Two useful descriptions of subspaces include:

1. Equations. We can describe a subspace $\mathcal{S}$ as a set of vectors satisfying certain linear relationships between their entries.

2. Generators. We can describe a subspace $\mathcal{S}$ as the span of a set of vectors.

For instance, the subspace $\mathcal{S}$ of 3-dimensional vectors whose third entry is the sum of the first and second entries can be expressed as either:

$$\mathcal{S} = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \; : \; v_1 + v_2 - v_3 = 0 \right\} \qquad \text{or} \qquad \mathcal{S} = \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Depending on the task, one description may be more convenient than the other. We can use Gaussian elimination to go from one description to the other.

1.4 Basis

We say $\{v_1, \ldots, v_k\}$ is a *basis* of a subspace $\mathcal{S}$ if they generate $\mathcal{S}$ and are LI.
In general, a subspace has infinitely many different bases. However, they all must have the same cardinality (i.e., the number of vectors in the basis) – this is called the *dimension* of the subspace.

2 Exercises

1. Note that the notions of linear independence and linear dependence are not quite symmetric. In particular:
   (a) Show that if $\{v_1, \ldots, v_k\}$ are LI, then any subset of the vectors is also LI.
   (b) Does a similar statement hold for linear dependence? Prove this, or give a counterexample.
2. Identify if the following sets of vectors are linearly independent or not.
   (a) $A = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \end{bmatrix} \right\}$.
   (b) $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.
   (c) $C = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$
   (d) $D = \left\{ \begin{bmatrix} 0 \\ -1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 11 \\ -6 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -12 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 1000 \\ 100 \\ 10 \\ 1 \end{bmatrix} \right\}$.
3. Let $A = \begin{bmatrix} -3 & 1 & 0 & 5 \\ -2 & 2 & -2 & 1 \\ 1 & -3 & 4 & 3 \end{bmatrix}$. Answer the following questions.
   (a) Are the columns of A linearly independent?
   (b) Find a set of generators for $N(A)$, the nullspace of A.
   (c) Find a basis of the column space $C(A)$, aka the span of columns.

   You can do it either by inspection, or algorithmically, using Gauss-Jordan elimination.
4. Consider the two subspaces

$$U = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} : v_1 + 2v_2 - 4v_3 = 0 \right\}$$

(a) Write a description of U as a set of vectors that satisfy linear relationships.

(b) Write a description of V as the span of a set of generators.

(c) Compute the dimension and a basis of U and V.

(d) Compute the dimension and a basis of $U \cap V$.

3 Solutions

1. (a) True. Suppose $\{v_1, \cdots, v_k\}$ are LI. Consider $\{v_1, \cdots, v_{k-1}\}$. Suppose there are coefficients $\alpha'_1, \cdots \alpha'_{k-1}$ with $\sum_{i=1}^{k-1} \alpha'_i v_i = 0$. Now choose $\alpha_i = \alpha'_i$ for all i from 1 to $k-1$ and set $\alpha_k = 0$. For this choice, we have $\sum_{i=1}^{k} \alpha_i v_i = 0$ and since the v_i's are LI, by definition, we must have that all the α_i's are zero which means all the α'_i's are zero too. This implies $\{v_1, \cdots, v_{k-1}\}$ are LI.

   (b) False. Consider, for example

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \text{ and } \begin{bmatrix}1\\1\end{bmatrix}$$

   It is easy to see that the vectors are LD. And yet the first two vectors are LI.

2. (a) Yes. Let's suppose there is a linear combination of the two vectors that results in $\mathbf{0}$. This means $\begin{bmatrix}0\\0\end{bmatrix} = \lambda_1 \begin{bmatrix}2\\3\end{bmatrix} + \lambda_2 \begin{bmatrix}5\\7\end{bmatrix} = \begin{bmatrix}2\lambda_1 + 5\lambda_2\\3\lambda_1 + 7\lambda_2\end{bmatrix}$. Therefore $2\lambda_1 + 5\lambda_2 = 0$ and $3\lambda_1 + 7\lambda_2 = 0$. We can then solve

$$\lambda_1 = 5 \cdot (3\lambda_1 + 7\lambda_2) - 7 \cdot (2\lambda_1 + 5\lambda_2) = 5 \cdot 0 - 7 \cdot 0 = 0$$
$$\lambda_2 = 3 \cdot (2\lambda_1 + 5\lambda_2) - 2 \cdot (3\lambda_1 + 7\lambda_2) = 3 \cdot 0 - 2 \cdot 0 = 0.$$

   (b) Yes. Let's suppose there is a linear combination of the three vectors that results in $\mathbf{0}$. This means $\begin{bmatrix}0\\0\\0\end{bmatrix} = \lambda_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + \lambda_2 \begin{bmatrix}1\\1\\0\end{bmatrix} + \lambda_3 \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}\lambda_1 + \lambda_2 + \lambda_3\\ \lambda_1 + \lambda_2\\ \lambda_1\end{bmatrix}$. Therefore $\lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 + \lambda_2 = \lambda_1 = 0$ which yields $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

   (c) No. There is a non-trivial linear combination that results in $\mathbf{0}$:

$$1 \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} - 2 \cdot \begin{bmatrix}4\\5\\6\end{bmatrix} + 1 \cdot \begin{bmatrix}7\\8\\9\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

   (d) No. There are 5 vectors in $\mathbb{R}^4$ which means they cannot be linearly independent.

3. (a) No, the columns of the matrix are not linearly independent. The third and fourth columns fall into the span of the first two columns. We perform row operations.

$$A = \begin{bmatrix}-3 & 1 & 0 & 5\\-2 & 2 & -2 & 1\\1 & -3 & 4 & 3\end{bmatrix} \xRightarrow{\text{swap } R_1 \& R_3} \begin{bmatrix}1 & -3 & 4 & 3\\-2 & 2 & -2 & 1\\-3 & 1 & 0 & 5\end{bmatrix}$$
$$\xRightarrow[R_3 \leftarrow R_3 + 3R_1]{R_2 \leftarrow R_2 + 2R_1} \begin{bmatrix}1 & -3 & 4 & 3\\0 & -4 & 6 & 7\\0 & -8 & 12 & 14\end{bmatrix} \xRightarrow{R_3 \leftarrow R_3 - 2R_2} \begin{bmatrix}1 & -3 & 4 & 3\\0 & -4 & 6 & 7\\0 & 0 & 0 & 0\end{bmatrix}$$
$$\xRightarrow{R_2 \leftarrow -R_2/4} \begin{bmatrix}1 & -3 & 4 & 3\\0 & 1 & -3/2 & -7/4\\0 & 0 & 0 & 0\end{bmatrix} \xRightarrow{R_1 \leftarrow R_1 + 3R_2} \begin{bmatrix}\color{red}{1} & 0 & -1/2 & -9/4\\0 & \color{red}{1} & -3/2 & -7/4\\0 & 0 & 0 & 0\end{bmatrix} = R$$

   Pivots of the rref matrix R is indicated in red. This tells us that the pivot columns are the first and the second column – which together will form a basis of $C(A)$. This is also the solution to part (c)

(b) We wish to solve for x the following matrix vector equation

$$\begin{bmatrix}1 & 0 & -1/2 & -9/4\\ 0 & 1 & -3/2 & -7/4\\ 0 & 0 & 0 & 0\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=0$$

We see that all solutions are of the form

$$(\frac{1}{2}x_3+\frac{9}{4}x_4, \frac{3}{2}x_3+\frac{7}{4}x_4, x_3, x_4) = x_3\cdot(\frac{1}{2},\frac{3}{2},1,0)+x_4(\frac{9}{4},\frac{7}{4},0,1)$$

Which implies $\{(\frac{1}{2},\frac{3}{2},1,0),(\frac{9}{4},\frac{7}{4},0,1)\}$ is the basis of the nullspace of A.

(c) Algorithmically. At first, $B=\emptyset$. Since B was empty, the first column $\begin{bmatrix}-3\\ -2\\ 1\end{bmatrix}$ is not in its span – so we add the first column to B. The second column is not a multiple of the first column so we add it to B as well. The third column, however, is in Span(B) since

$$\begin{bmatrix}0\\ -2\\ 4\end{bmatrix}=(-1/2)\cdot\begin{bmatrix}-3\\ -2\\ 1\end{bmatrix}+(-3/2)\cdot\begin{bmatrix}1\\ 2\\ -3\end{bmatrix}$$

which means it is *not* added to B. The forth column is neither added to B since

$$\begin{bmatrix}5\\ 1\\ 3\end{bmatrix}=(-9/4)\cdot\begin{bmatrix}-3\\ -2\\ 1\end{bmatrix}+(-7/4)\cdot\begin{bmatrix}1\\ 2\\ -3\end{bmatrix}.$$

This concludes that the basis of $C(A)$ consists of the first two columns of A which are

$$\left\{\begin{bmatrix}-3\\ -2\\ 1\end{bmatrix},\begin{bmatrix}1\\ 2\\ -3\end{bmatrix}\right\}.$$

On the other hand, we can do it by performing row operations.

$$A=\begin{bmatrix}-3 & 1 & 0 & 5\\ -2 & 2 & -2 & 1\\ 1 & -3 & 4 & 3\end{bmatrix}\xRightarrow{\text{swap } R_1\&R_3}\begin{bmatrix}1 & -3 & 4 & 3\\ -2 & 2 & -2 & 1\\ -3 & 1 & 0 & 5\end{bmatrix}$$

$$\xRightarrow[R_3\leftarrow R_3+3R_1]{R_2\leftarrow R_2+2R_1}\begin{bmatrix}1 & -3 & 4 & 3\\ 0 & -4 & 6 & 7\\ 0 & -8 & 12 & 14\end{bmatrix}\xRightarrow{R_3\leftarrow R_3-2R_2}\begin{bmatrix}1 & -3 & 4 & 3\\ 0 & -4 & 6 & 7\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\xRightarrow{R_2\leftarrow -R_2/4}\begin{bmatrix}1 & -3 & 4 & 3\\ 0 & 1 & -3/2 & -7/4\\ 0 & 0 & 0 & 0\end{bmatrix}\xRightarrow{R_1\leftarrow R_1+3R_2}\begin{bmatrix}{\color{red}1} & 0 & -1/2 & -9/4\\ 0 & {\color{red}1} & -3/2 & -7/4\\ 0 & 0 & 0 & 0\end{bmatrix}=R$$

Pivots of the rref matrix R is indicated in red. This tells us that the pivot columns are the first and the second column – which together will form a basis of $C(A)$.

4. (a) We first notice that the
$$\begin{bmatrix}1\\3\\1\\2\end{bmatrix} = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$$
which means we can remove the third vector off the collection without altering the span. Thus, U is a set of vectors in forms of $x\begin{bmatrix}1\\1\\1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}x\\x+y\\x\\y\end{bmatrix}$. In other words,
$$U = \left\{ \begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix} : u_1 = u_3, u_1 + u_4 = u_2 \right\}.$$

(b) With $v_1 + 2v_2 - 4v_3 = 0$, we can write
$$\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \begin{bmatrix}-2v_2+4v_3\\v_2\\v_3\\v_4\end{bmatrix} = v_2\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + v_3\begin{bmatrix}4\\0\\1\\0\end{bmatrix} + v_4\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$
which tells us that
$$V = \operatorname{Span}\left\{ \begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}4\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}.$$

(c) In part a, we eliminated the third vector and the two remaining vectors are linearly independent. This means U's dimension is 2, and thus U's basis is
$$\left\{ \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix} \right\}.$$
In part b, we found a generator of V with 3 vectors. Moreover, they are linearly independent. This tells us that V's dimension is 3, and thus V's basis is
$$\left\{ \begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}4\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}.$$

(d) Any vector $t = \begin{bmatrix}t_1\\t_2\\t_3\\t_4\end{bmatrix} \in U \cap V$ must satisfy the equations describing of both U and V. This means $t_1 = t_3$, $t_1 + t_4 = t_2$, and $t_1 + 2t_2 - 4t_3 = 0$. If we fix t_1, we will have $t_3 = t_1$, $t_2 = (4t_3 - t_1)/2 = 3t_1/2$, and $t_4 = t_2 - t_1 = 3t_1/2 - t_1 = t_1/2$. Therefore,

$$t = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} t_1 \\ 3t_1/2 \\ t_1 \\ t_1/2 \end{bmatrix} = (t_1/2) \cdot \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

which implies $U \cap V = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right\}$. Moreover, this tells us that $U \cap V$ has dimension 1 with a basis $\left\{ \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right\}$.