Linear Algebra and Optimization
MIT Fall 2022

Recitation 5

Thursday September 22, 2022

1 Recap

1.1 Vector Space

A vector space V is a set that is closed under **vector addition** and **scalar multiplication**. These two operations must satisfy a set of axioms[1]. A basic example is the real vector space $\mathbb{R}^n$, where

- every vector is represented by a list of n real numbers
- scalars are real numbers
- addition is defined element-wise
- scalar multiplication is multiplication on each term separately

For a general vector space, the scalars are elements of a field F (e.g., the complex numbers), in which case V is called a vector space over F. If $W \subseteq V$ is also a vector space with respect to the operations in V, then W is called a subspace of V.

Key Fact. If S_1 and S_2 both are vector space, then $S = S_1 \cap S_2$ is a subspace.

1.2 Column Space

The **column space** of an $m \times n$ matrix A, denoted $C(A)$, is the set of all **linear combinations** of columns of A, or the **span** of A.

- $C(A) = \{Ax \mid x \in \mathbb{R}^n\}$.
- $Ax = b$ has a solution iff $b \in C(A)$.
- If $m = n$, then A is invertible iff $C(A) = \mathbb{R}^n$.

1.3 Null Space

The **null space** of A, denoted $N(A)$, is the set of vectors x such that $Ax = 0$.

- $N(A) = \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\}$.
- If B is a square and invertible matrix, then $N(A) = N(BA)$.
- If A is $n \times n$, then $C(A) = \mathbb{R}^n \iff N(A) = \{\mathbf{0}\} \iff A$ is invertible.

[1] https://en.wikipedia.org/wiki/Vector$_{spaceDefinition, ormoreformallyathttps}$: //encyclopediaofmath.org/wiki/V

2 Exercises

1. Is V a real vector space?
   (a) V is the set of all n-dimensional vectors with positive entries, with usual vector operations.
   (b) V is the set of all n-dimensional vectors whose elements sum to 0, with usual vector operations.
   (c) V is the set of all $n \times n$ diagonal matrices, with usual matrix operations.
   (d) V is the set of all polynomials with degree up to d, with usual polynomial operations.
   (e) V is the set of all constant functions, i.e. $f(x) = c$ for some constant $c \in \mathbb{R}$, with usual real number operations.
   (f) V is the set of all single-variable polynomial whose value at 0 is 1, i.e. polynomial P with $P(0) = 1$, with usual polynomial operations.
2. True or False
   (a) Define the row space of matrix A as the span of the row vectors of A. If A is a square matrix, then the row space of A equals the column space.
   (b) The row space of A is equal to the column space of A^T.
   (c) If the row space of A equals the column space, then $A^T = A$.
   (d) A 4 by 4 permutation matrix has column space equal to $\mathbb{R}^4$.
   (e) Let $v \in N(A)$. If x is a solution to equation $A\mathbf{x} = b$, so is $x + v$.
3. A is a 3 by 3 matrix.
   (a) $x = [1\ 2\ 3]^T$ is in the null space of some matrix A. What is $A(2x)$?
   (b) $y = [1\ 2\ 4]^T$ is also in the null space of A. What is $A(2x + y)$?
   (c) $Az = [1\ 2\ 5]^T$. What is $A(2x + y + z)$
4. $Ax = b$, A is a 3 by 3 matrix, x and b are 3-dimensional vectors.
   (a) When $b = [1\ 2\ 3]^T$ we have infinite solutions. Is b in the column space of A?
   (b) Assume (a) is still true. Is A invertible?
   (c) Now assume A is invertible, do we have a solution when $b = \mathbf{0}$? How many?
5. $$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$
   (a) Find a nonzero vector in the null space of A, if any exists.
   (b) Is A invertible?
   (c) Find a vector in the column space of A that is not equal to either column.
   (d) Find a vector in the row space of A that is not equal to either row.

3 Solutions

1. (a) No. Take $v = \begin{bmatrix}1 & 1 & ... & 1\end{bmatrix}^T$. All entries of v is positive so $v \in V$. However, $(-1) \cdot v = \begin{bmatrix}-1 & -1 & ... & -1\end{bmatrix}^T$ has negative entries; thus is *not* in V.

   (b) Yes. If v has elements sum to 0, then λv has elements sum to $\lambda \cdot 0 = 0$. If we have another vector u whose elements also sum to 0, then $u + v$ has elements sum to u's sum + v's sum $= 0 + 0 = 0$.

   (c) Yes. The sum of two diagonal matrices is a diagonal matrix. The scalar multiple of a diagonal matrix is also a diagonal matrix.

   (d) Yes. The sum of two up-to-degree-d polynomials also has degree at most d. The scalar multiple does not change the degree – so it is still at most d.

   (e) Yes. For any two constant functions $f(x) = c_1, g(x) = c_2$, and any real numbers a, b, $af(x) = ac_1$ is a constant function; $f(x) + g(x) = c_1 + c_2$ is a constant function.

   (f) No. If we have a polynomial P which $P(0) = 1$, then $Q = 5P$ has $Q(0) = 5 \cdot P(0) = 5(1) = 5 \neq 1$, which violates scalar multiplicity condition. Moreover, if we have two polynomials P, Q with $P(0) = Q(0) = 1$, then $(P + Q)(0) = P(0) + Q(0) = 1 + 1 = 2 \neq 1$ which violates the vector addition condition.

2. (a) False. The 2 by 2 matrix A in question 5 is a counter example.

   (b) True. The set of rows of A is identical to the set of columns of A^T.

   (c) False. Counter example: $A = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$. Both the column and row space are equal to $\mathbb{R}^2$, but $A \neq A^T$.

   (d) True. Any permutation matrix is invertible which implies that its span is $\mathbb{R}^4$.

   (e) $v \in N(A)$ means $Av = \mathbf{0}$. As x is a solution to $A\mathbf{x} = b$, we have $Ax = b$. This gives us $A(x + v) = Ax + Av = b + \mathbf{0} = b$ which means $x + v$ is also an answer.

3. (a) $x \in A$ means $Ax = \mathbf{0}$. This gives $A(2x) = 2(Ax) = 2(\mathbf{0}) = \mathbf{0}$.

   (b) We also have $Ay = \mathbf{0}$. This gives $A(2x + y) = 2(Ax) + Ay = 2(\mathbf{0}) + \mathbf{0} = \mathbf{0}$.

   (c) $A(2x + y + z) = A(2x + y) + Az = \mathbf{0} + Az = Az = [125]^T$

4. (a) Take one solution x that gives $Ax = b$. This means $b \in C(A)$ as we can right multiply A by some vector (x in this case) to get b.

   (b) A is *not* invertible. Take two distinct answers u, v of $Ax = b$. This gives $Au = Av = b$ so $A(u - v) = \mathbf{0}$ which implies $u - v \in N(A)$. In addition, $u - v \neq \mathbf{0}$ so $N(A)$ is non-trivial – this means A is not invertible.

   (c) If A is invertible, we can solve $Ax = \mathbf{0}$ by left-multiplying both sides by A^{-1} which gives $x = \mathbf{0}$.

5. (a) $\begin{bmatrix}2\\ -1\end{bmatrix}$. We notice that at every row, the second column is twice of the first column. This means We see that $A\begin{bmatrix}2\\ -1\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$.

(b) Per part a), we see that $A\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ which means $\begin{bmatrix}2\\-1\end{bmatrix}\in N(A)$. This tells us that $N(A)$ is non-trivial; thus A is *not* invertible.

(c) Any vector in A's column space can be expressed as $x\cdot\begin{bmatrix}1\\3\end{bmatrix}+y\cdot\begin{bmatrix}2\\6\end{bmatrix}=(x+2y)\cdot\begin{bmatrix}1\\3\end{bmatrix}$. In other words, any scalar multiple of $\begin{bmatrix}1\\3\end{bmatrix}$ is in A's column space.

(d) Any vector in A's row space can be expressed as $x\cdot\begin{bmatrix}1\\2\end{bmatrix}+y\cdot\begin{bmatrix}3\\6\end{bmatrix}=(x+3y)\cdot\begin{bmatrix}1\\2\end{bmatrix}$. In other words, any scalar multiple of $\begin{bmatrix}1\\2\end{bmatrix}$ is in A's row space.