Recitation 7

Thursday September 30, 2022

1 Recap

1.1 Two Descriptions of Subspaces

Two useful descriptions of subspaces are:

1. Equations. We can describe a subspace $\mathcal{S}$ as a set of vectors satisfying certain linear relationships between their entries.

2. Generators. We can describe a subspace $\mathcal{S}$ as the span of a set of vectors.

For instance, the subspace $\mathcal{S}$ of 3-dimensional vectors whose third entry is the sum of the first and second entries can be expressed as either:

$$\mathcal{S} = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \ : \ v_1 + v_2 - v_3 = 0 \right\} \qquad \text{or} \qquad \mathcal{S} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Depending on the task, one description may be more convenient than the other. We can use Gaussian elimination to go from one description to the other.

1.2 Translating Between Two Descriptions

Here we describe in detail how to algorithmically convert one kind of description into the other. Please make sure that you understand why these steps work – this is essentially the method that you've already learned, that uses Gaussian elimination and parametrizes the solution using "free variables."

1.2.1 Equations to Generators

Let $\mathcal{S}$ be a subspace of n-dimensional vectors x satisfying a system of m linear equations. We proceed as follows:

1. Write the system as a matrix-vector product $Ax = \mathbf{0}$ where A is an $m \times n$ coefficient matrix. This means that the subspace $\mathcal{S}$ is the nullspace of A.

2. Perform Gauss-Jordan elimination on A to obtain the rref matrix R.

3. Permute the columns of R so that it is in the form $\begin{bmatrix} I_r & P \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ where $r = \text{rank}(A)$, P is an $r \times (n - r)$ matrix, and $\mathbf{0}$'s are zero matrices with proper dimensions.

   We also need to record the permutation of columns π.

4. Define $Q = \begin{bmatrix} -P \\ I_{n-r} \end{bmatrix}$.

5. Permute rows of Q with π^{-1}. The columns of the resulting matrix B then form a basis of $\mathcal{S}$. In other words, the columns of B are linearly independent, and span $\mathcal{S}$.

Example 1. Find a generator description of

$$\mathcal{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \; : \; x_1 + 2x_2 - 2x_3 - x_4 = 0 \text{ and } 2x_1 + 4x_2 - 3x_3 + 2x_4 = 0 \right\}.$$

To do so, we follow the procedure.

1. We form the system $Ax = \mathbf{0}$ where $A = \begin{bmatrix} 1 & 2 & -2 & -1 \\ 2 & 4 & -3 & 2 \end{bmatrix}$.

2. Do Gauss-Jordan elimination to put A in rref
$$A = \begin{bmatrix} 1 & 2 & -2 & -1 \\ 2 & 4 & -3 & 2 \end{bmatrix} \qquad \xRightarrow{G-J} \qquad \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 0 & 1 & 4 \end{bmatrix} = R.$$

3. Permute the columns of R
$$R = \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xRightarrow{\pi(1,2,3,4)=(1,3,2,4)} \begin{bmatrix} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} I_2 & P \end{bmatrix} \text{ where } P = \begin{bmatrix} 2 & 7 \\ 0 & 4 \end{bmatrix}.$$

4. Derive
$$Q = \begin{bmatrix} -P \\ I_2 \end{bmatrix} = \begin{bmatrix} -2 & -7 \\ 0 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

5. Permute Q's rows with respect to π^{-1} to derive B
$$Q = \begin{bmatrix} -2 & -7 \\ 0 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \xRightarrow{\pi^{-1}} \begin{bmatrix} -2 & -7 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{bmatrix} = B.$$

This tells us that
$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\} \text{ is a basis of } \mathcal{S}.$$

We can easily verify that the output is correct, i.e., these vectors are linearly independent and span $\mathcal{S}$.

1.2.2 Generators to Equations

Now suppose that we have a subspace $\mathcal{S}$ described by a given set of generators $\{v_1, \ldots, v_k\}$. To find a (minimal) equation description of $\mathcal{S}$, we "transpose" the procedure given in section 1.2.1. The slight changes include

1. We now set $A = \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix}^T$.
2. Each column of the matrix B (in the final step) corresponds to the coefficients of a linear equation.

The obtained linear equations are LI, and their solution set is equal to $\mathcal{S}$.

Example 2. Find an equation description of

$$\mathcal{S} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 8 \end{bmatrix} \right\}.$$

To do so, we follow the procedure.

1. Initialize $A = \begin{bmatrix} 1 & 0 & 3 & 5 \\ 2 & 1 & 5 & 8 \\ 1 & 2 & 2 & 0 \\ 1 & 0 & 0 & 8 \end{bmatrix}$.

2. Do Gauss-Jordan elimination to put A in rref

$$A = \begin{bmatrix} 1 & 0 & 3 & 5 \\ 2 & 1 & 5 & 8 \\ 1 & 2 & 2 & 0 \\ 1 & 0 & 0 & 8 \end{bmatrix} \xRightarrow{G-J} \begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

3. No permutation of R's columns is needed since it is already in the form we want. This means

$$R = \begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xRightarrow{\text{no permutations}} \begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & P \\ 0 & 0 \end{bmatrix} \text{ where } P = \begin{bmatrix} 8 \\ -3 \\ -1 \end{bmatrix}.$$

4. Define

$$Q = \begin{bmatrix} -P \\ I_1 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \\ 1 \\ 1 \end{bmatrix}.$$

5. Since we didn't permute anything in step 3, we do not need to permute the rows of Q. This means

$$Q = \begin{bmatrix} -8 \\ 3 \\ 1 \\ 1 \end{bmatrix} \xRightarrow{\text{no permutations}} \begin{bmatrix} -8 \\ 3 \\ 1 \\ 1 \end{bmatrix} = B.$$

This tells us that

$$\mathcal{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : -8x_1 + 3x_2 + x_3 + x_4 = 0 \right\}.$$

So in this case $\mathcal{S}$ is a 3-dimensional subspace in $\mathbb{R}^4$. You can easily verify that it contains the four given generators (and therefore, their span).

1.3 Tall Matrix

Let A be an $m \times n$ matrix with $m \geq n$. Then, the following statements are equivalent.

1. The columns of A are linearly independent.
2. If $Ax = b$ is solvable, then the solution is unique.
3. A has a left inverse.
4. $N(A) = \{\mathbf{0}\}$.
5. $\text{rank}(A) = n$.

1.4 Wide Matrix

Let A be an $m \times n$ matrix with $m \leq n$. Then, the following statements are equivalent.

1. The rows of A are linearly independent.
2. $Ax = b$ is solvable for any $b \in \mathbb{R}^m$.
3. A has a right inverse.
4. $C(A) = \mathbb{R}^m$.
5. $\text{Rank}(A) = m$.

1.5 Square Matrix

Let A be an $n \times n$ square matrix. Then, the following statements are equivalent.

1. A is invertible, i.e. A^{-1} exists.
2. $Ax = b$ is uniquely solvable for any $b \in \mathbb{R}^n$.
3. $N(A) = \{\mathbf{0}\}$.
4. $C(A) = \mathbb{R}^n$.
5. $\text{Rank}(A) = n$.

1.6 Rank-Nullity Theorem

Let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ matrix. Then we have $\text{rank}(A) + \dim N(A) = n$.

2 Exercises

1. Denote subspaces

$$U = \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} : u_1 - u_3 + 2u_4 = 0,\ u_2 - u_3 + 3u_4 = 0,\ u_1 + u_2 - 2u_3 + 5u_4 = 0 \right\}$$

$$V = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ -4 \end{bmatrix} \right\}.$$

   (a) Write a description of U as a set of vectors that satisfy linear relationships.

   (b) Write a description of V as the span of a set of generators.

2. Let A and B be $n \times n$ matrices. For any sets P and Q, denote $P + Q = \{p + q \mid p \in P, q \in Q\}$. Determine whether the following statements are true or false.

   (a) $C(A + B) = C(A) + C(B)$

   (b) $N(A + B) = N(A) + N(B)$.

   (c) If $x \in C(A)$ and $x \in C(B)$, then $x \in C(A + B)$.

   (d) If $x \in N(A)$ and $x \in N(B)$, then $x \in N(A + B)$.

   (e) If $N(A) \cap C(B) \neq \{\mathbf{0}\}$, then AB is not invertible.

   (f) If $N(A) \supseteq C(B)$ then $AB = 0$.

   (g) A tall matrix always has a left inverse.

   (h) A square matrix A is *diagonal* if $a_{ij} = 0$ when $i \neq j$. The rank of a diagonal matrix is the number of nonzero entries.

   (i) If P is an $n \times n$ permutation matrix, then $N(P) = \mathbb{R}^n$.

3. Alice and Bob try to describe a d-dimensional subspace $\mathcal{S} \subseteq \mathbb{R}^n$ in two different ways.

   (a) Alice describes $\mathcal{S}$ as the span of linearly independent vectors. Alice discovers that no matter how she comes up with different set of linearly independent vectors, the number of vectors never changes. Should Alice be surprised by this result? Why, or why not?

   (b) Bob describes $\mathcal{S}$ as a set of vectors that satisfy a certain number of linearly independent equations (i.e. no equation can be written as a linear combination of the remaining ones.) Bob discovers that no matter how he comes up with different sets of linearly independent equations, the number of equations never changes. Should Bob be surprised by this result? Why, or why not?

   (c) Alice and Bob finally meet up and share their discoveries. They share with one another the number of vectors (by Alice) and the number of equations (by Bob). Then they are stunned by the fact that the two numbers always add up to the same constant. Should Alice and Bob be surprised by this result? What is this mysterious constant, in terms of n and d?

3 Solutions

1. (a)

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \\ 1 & 1 & -2 & 5 \end{bmatrix} \xRightarrow{G-J} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & P \\ 0 & 0 \end{bmatrix} \text{ where } P = \begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}$$

This gives

$$Q = B = \begin{bmatrix} -P \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ which means } U = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b)

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 2 & -4 & 3 & 1 \\ 1 & -2 & 0 & -4 \end{bmatrix} \xRightarrow{G-J} \begin{bmatrix} 1 & -2 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xRightarrow{\pi(1,2,3,4)=(1,3,2,4)} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & P \\ 0 & 0 \end{bmatrix} \text{ where } P = \begin{bmatrix} -2 & -4 \\ 0 & 3 \end{bmatrix}$$

This gives

$$Q = \begin{bmatrix} -P \\ I_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \xRightarrow{\pi^{-1}} \begin{bmatrix} 2 & 4 \\ 1 & 0 \\ 0 & -3 \\ 0 & 1 \end{bmatrix}$$

which means

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} : 2v_1 + v_2 = 0,\ 4v_1 - 3v_3 + v_4 = 0 \right\}.$$

2. (a) False. A counterexample is in $\mathbb{R}$ when $A = \begin{bmatrix} 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \end{bmatrix}$. Then $C(A) = C(B) = \mathbb{R}$ which makes $C(A) + C(B) = \mathbb{R}$. However, $A + B = \begin{bmatrix} 0 \end{bmatrix}$ whose column space is $\{0\}$.

   (b) False. A counterexample is in $\mathbb{R}$ when $A = \begin{bmatrix} 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \end{bmatrix}$. Then $N(A) = N(B) = \{0\}$ which makes $N(A) + N(B) = \{0\}$. However, $A + B = \begin{bmatrix} 0 \end{bmatrix}$ whose null space is $\mathbb{R}$.

   (c) False. A counterexample is in $\mathbb{R}$ when $A = \begin{bmatrix} 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 \end{bmatrix}$, and $x = 1$. We see that $C(A) = C(B) = \mathbb{R}$ so $x \in C(A)$ and $x \in C(B)$. However, $A + B = \begin{bmatrix} 0 \end{bmatrix}$ whose column space does not contain $x = 1$.

   (d) True. If $x \in N(A)$ and $x \in N(B)$, then $Ax = \mathbf{0}$ and $Bx = \mathbf{0}$. Then we have $(A + B)x = Ax + bx = \mathbf{0} + \mathbf{0} = \mathbf{0}$ which means $x \in N(A + B)$.

(e) True. If $N(A) \cap C(B) \neq \{\mathbf{0}\}$, then there must exists a non-zero vector $v \in N(A) \cap C(B)$. This means $v \in N(A)$ and $v \in C(B)$. So $Av = \mathbf{0}$ and there must exist another vector u which $Bu = v$. Note that if $u = \mathbf{0}$, then $v = \mathbf{0}$ which is contradictory; therefore $u \neq \mathbf{0}$. So we have $(AB)u = A(Bu) = Av = \mathbf{0}$ and $u \neq \mathbf{0}$. This implies $N(AB) \neq \{\mathbf{0}\}$ which means AB is not invertible.

(f) True. For any vector x we have $Bx \in C(B) \subseteq N(A)$, and thus $ABx = 0$, which implies that AB is the zero matrix.

(g) False. A simple counterexample is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This is a tall matrix, but it has no left- (or right-) inverse.

(h) True. As long as they are nonzero, the column (or row) vectors are linearly independent.

(i) False. If P is a permutation matrix, then $N(P) = \{0\}$, since $Px = 0$ implies $x = 0$.

3. (a) Alice describes $\mathcal{S}$ as a set of vectors that a set of linearly independent vectors $\mathcal{V}$ generates. In other words, $\mathcal{V}$ is a *basis* of $\mathcal{S}$. Although there are infinitely many set of bases of $\mathbb{S}$, all of them must have the same cardinality (equal to $\dim(\mathcal{S}) = d$) which means $|\mathcal{V}|$ never changes.

For simplicity, let's say $|\mathcal{V}| = d$. Take a basis $\mathcal{V} = \{v_1, ..., v_d\}$ of $\mathcal{S}$ and construct a $d \times n$ matrix $A = \begin{bmatrix} v_1 & ... & v_d \end{bmatrix}^T$. This also tells us that $\operatorname{rank}(A) = d$.

(b) Let a set of coefficients of those linearly independent equations be $c_1, c_2, ..., c_k$. We claim that $N(A) = \operatorname{Span}\{c_1, ..., c_k\}$. To see why it's true, let's take a vector $x \in N(A)$. This means $\mathbf{0} = Ax = \begin{bmatrix} v_1^T x & ... & v_d^T x \end{bmatrix}$ which implies $\langle v_1, x \rangle = ... = \langle v_d, x \rangle = 0$. In other words, an equation with coefficients x is satisfied by $v_1, ..., v_d$ and thus by any vector in $\mathcal{S}$. Since $\{c_1, ..., c_k\}$ is sufficient for describing $\mathcal{S}$, then x must be a linear combination of $\{c_1, ..., c_k\}$. This proves that $N(A) = \operatorname{Span}\{c_1, ..., c_k\}$.

Furthermore, we know that $\{c_1, ..., c_k\}$ are linearly independent, which means it is a basis of $N(A)$. Although there are infinitely many set of bases of $N(A)$, all of them must have the same cardinality (equal to $\dim N(A) = k$) which means the number of linearly independent equations k never changes.

(c) Alice has number $d = \operatorname{rank}(A)$ and Bob has number $k = \dim N(A)$. Their sum is

$$d + k = \operatorname{rank}(A) + \dim N(A) = \text{the number of columns of } A = n$$

due to the Rank-Nullity Theorem. Therefore, summing up to n is not coincidence.