Linear Algebra and Optimization
MIT Fall 2021

Recitation 9

Thursday October 6, 2022

1 Recap

1.1 Determinant

1.1.1 Algebraic View

The *determinant* of a square matrix $A \in \mathbb{R}^{n\times n}$ is defined as

$$\det A = \sum_{\sigma} \left((-1)^{\text{sign}(\sigma)} \cdot \prod_{i=1}^{n} A_{i,\sigma(i)} \right) \tag{1}$$

where σ iterates over any permutation of $\{1, ..., n\}$ and $\text{sign}(\sigma)$ is the parity of σ.
An equivalent definition (more computationally convenient) for the determinant is

$$\det A = (\pm 1) \cdot (\text{product of pivots in ref}(A)),$$

where the sign depends on the parity of the number of row exchanges in the REF.

1.1.2 Geometric View

The *determinant* of a square matrix $A \in \mathbb{R}^{n\times n}$ is the scaling factor between $vol(S)$ and $vol(\phi_A(S))$ taking into account handedness; where $\phi_A(S) = \{Ax : x \in S\}$ is the region which ϕ_A maps S into.

1.1.3 Properties

Let A, B be $n \times n$ matrices. Let C be another square matrix and D be a matrix with proper dimension.

1. Swapping two rows (or two columns) of A *negates* the determinant.
2. Adding a row to another row *does not* change the determinant.
3. Adding a column to another column *does not* change the determinant.
4. Multiplying a row/column by a scalar c *changes* the determinant by a factor of c.
5. $\det A^T = \det A$
6. $\det AB = \det A \cdot \det B$
7. $\det(A + B) \neq \det A + \det B$
8. $\det \left(\begin{bmatrix} A & D \\ 0 & C \end{bmatrix} \right) = \det A \cdot \det C$.

1.2 Square Matrices Revisited

Let A be an $n \times n$ square matrix. Then, the following statements are equivalent.

1. A is invertible, i.e. A^{-1} exists
2. A has both a left inverse and a right inverse
3. The columns of A are linearly independent
4. The rows of A are linearly independent
5. $Ax = b$ is uniquely solvable for every $b \in \mathbb{R}^n$
6. $N(A) = \{\mathbf{0}\}$
7. $C(A) = \mathbb{R}^n$
8. $\text{Rank}(A) = n$
9. $\det A \neq 0$

1.3 Projection

Suppose that we want to find the orthogonal projection of a given vector w onto a k-dimensional subspace $\mathcal{V}$. In other words, we want to find a vector $v = \text{proj}_{\mathcal{V}} w$ for which 1) $v \in \mathcal{V}$, and 2) $(w - v) \perp u$ for all $u \in \mathcal{V}$.
Suppose that $\{v_1, \dots, v_k\}$ is an orthonormal basis of $\mathcal{V}$. Then,

$$\begin{aligned} v = \text{proj}_{\mathcal{V}} w &= \sum_{i=1}^{k} (w \cdot v_i) v_i = \sum_{i=1}^{k} v_i \cdot (v_i \cdot w) = \sum_{i=1}^{k} v_i \cdot (v_i^\top w) \\ &= \sum_{i=1}^{k} (v_i v_i^\top) w = \left(\sum_{i=1}^{k} v_i v_i^\top \right) w = Pw \end{aligned}$$

when $P = \sum_{i=1}^{k} v_i v_i^\top$ is an $n \times n$ matrix. The matrix P is the *orthogonal projection matrix* onto the subspace $\mathcal{V}$, and can also be written as

$$P = VV^\top, \quad \text{where } V = \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} \in \mathbb{R}^{n \times k}.$$

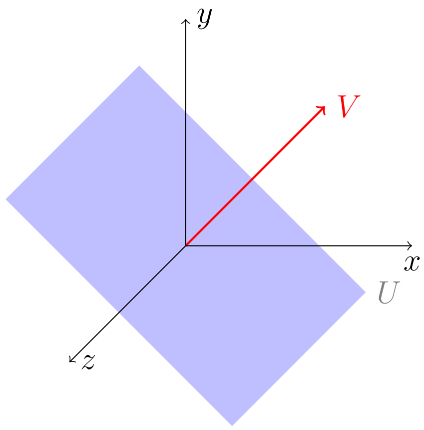
Note that P is symmetric (i.e. $P^\top = P$) and $\text{rank}(P) = \text{rank}(V) = k$.
Another interesting property of V is that $V^\top V = I_k$. Indeed, this is a necessary and sufficient condition for $\{v_1, \dots, v_k\}$ to be orthonormal.

2 Exercises

1. Use determinant properties to show that if A and B are square matrices such that AB is invertible, then both A and B are invertible.
2. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & \alpha & -5 \\ -1 & -1 & 2 \end{bmatrix}$. What values of α makes A *not* invertible?

3. Consider the two subspaces U and W as follows.

$$U = \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} : u_1 - u_2 = 0, u_1 - u_3 = 0 \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$
$$W = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} : w_1 + w_2 + w_3 = 0 \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$



The two subspaces are orthogonal complements (check!). It can be easily seen that the given set of generators bases are actually bases (why?).

(a) Find an orthonormal basis of U. Use it to derive the projection matrix P_U which projects vectors onto the subspace U.

(b) Find an orthonormal basis of W. Use it to derive the projection matrix P_W which projects vectors onto the subspace W.

(c) Notice that $P_U + P_W = I_3$. It turns out that this is not a coincidence. For any orthogonal complement subspaces $U, W \subseteq \mathbb{R}^n$, the sum of their corresponding projection matrices is exactly I_n. Can you explain why it is always the case? *Hint: Use orthogonal decomposition. What exactly is each component of the decomposition?*

4. Recall in 2D plane, a mirror matrix M is the matrix for which given a vector x, then Mx is the mirror image of x across a line L that passes through the origin.

(a) What can we tell about M^2 using the following fact: if we reflect a vector v across the line L *twice*, we end up with the original vector v.

(b) Using the answer from part (a), find the possible values of $\det M$.

(c) Take any 2-dimensional region $\mathcal{S}$ you like. What is the region that is produced from left-multiplying any $x \in \mathcal{S}$ by M? In other words, what is the region $\phi_M(\mathcal{S}) = \{Mx : x \in \mathcal{S}\}$?

(d) What is the volume of $\phi_M(\mathcal{S})$? How is to compared to the volume of $\mathcal{S}$? What can we tell about $\det M$?

(e) Recall from problem set 2 that if L makes angle θ with the x-axis, then we have $M = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$. Is the expression consistent with part a and d ?