Linear Algebra and Optimization
MIT Fall 2021

Recitation 9

Thursday October 6, 2022

1 Recap

1.1 Determinant

1.1.1 Algebraic View

The *determinant* of a square matrix $A \in \mathbb{R}^{n\times n}$ is defined as

$$\det A = \sum_{\sigma} \left((-1)^{\text{sign}(\sigma)} \cdot \prod_{i=1}^{n} A_{i,\sigma(i)} \right) \tag{1}$$

where σ iterates over any permutation of $\{1, ..., n\}$ and $\text{sign}(\sigma)$ is the parity of σ.
An equivalent definition (more computationally convenient) for the determinant is

$$\det A = (\pm 1) \cdot (\text{product of pivots in ref}(A)),$$

where the sign depends on the parity of the number of row exchanges in the REF.

1.1.2 Geometric View

The *determinant* of a square matrix $A \in \mathbb{R}^{n\times n}$ is the scaling factor between $vol(S)$ and $vol(\phi_A(S))$ taking into account handedness; where $\phi_A(S) = \{Ax : x \in S\}$ is the region which ϕ_A maps S into.

1.1.3 Properties

Let A, B be $n \times n$ matrices. Let C be another square matrix and D be a matrix with proper dimension.

1. Swapping two rows (or two columns) of A *negates* the determinant.
2. Adding a row to another row *does not* change the determinant.
3. Adding a column to another column *does not* change the determinant.
4. Multiplying a row/column by a scalar c *changes* the determinant by a factor of c.
5. $\det A^T = \det A$
6. $\det AB = \det A \cdot \det B$
7. $\det(A + B) \neq \det A + \det B$
8. $\det\left(\begin{bmatrix} A & D \\ 0 & C \end{bmatrix}\right) = \det A \cdot \det C$.

1.2 Square Matrices Revisited

Let A be an $n \times n$ square matrix. Then, the following statements are equivalent.

1. A is invertible, i.e. A^{-1} exists
2. A has both a left inverse and a right inverse
3. The columns of A are linearly independent
4. The rows of A are linearly independent
5. $Ax = b$ is uniquely solvable for every $b \in \mathbb{R}^n$
6. $N(A) = \{\mathbf{0}\}$
7. $C(A) = \mathbb{R}^n$
8. $\text{Rank}(A) = n$
9. $\det A \neq 0$

1.3 Projection

Suppose that we want to find the orthogonal projection of a given vector w onto a k-dimensional subspace $\mathcal{V}$. In other words, we want to find a vector $v = \text{proj}_{\mathcal{V}} w$ for which 1) $v \in \mathcal{V}$, and 2) $(w - v) \perp u$ for all $u \in \mathcal{V}$.
Suppose that $\{v_1, \ldots, v_k\}$ is an orthonormal basis of $\mathcal{V}$. Then,

$$\begin{aligned} v = \text{proj}_{\mathcal{V}} w &= \sum_{i=1}^{k} (w \cdot v_i) v_i = \sum_{i=1}^{k} v_i \cdot (v_i \cdot w) = \sum_{i=1}^{k} v_i \cdot (v_i^\top w) \\ &= \sum_{i=1}^{k} (v_i v_i^\top) w = \left(\sum_{i=1}^{k} v_i v_i^\top \right) w = Pw \end{aligned}$$

when $P = \sum_{i=1}^{k} v_i v_i^\top$ is an $n \times n$ matrix. The matrix P is the *orthogonal projection matrix* onto the subspace $\mathcal{V}$, and can also be written as

$$P = VV^\top, \quad \text{where } V = \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} \in \mathbb{R}^{n \times k}.$$

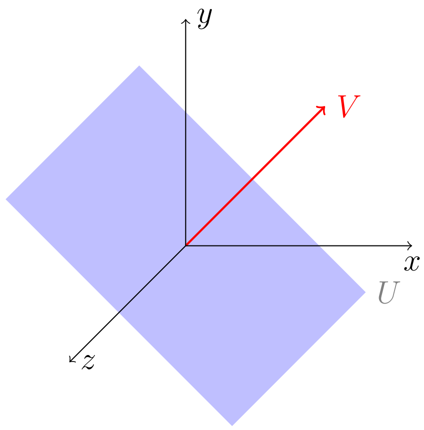
Note that P is symmetric (i.e. $P^\top = P$) and $\text{rank}(P) = \text{rank}(V) = k$.
Another interesting property of V is that $V^\top V = I_k$. Indeed, this is a necessary and sufficient condition for $\{v_1, \ldots, v_k\}$ to be orthonormal.

2 Exercises

1. Use determinant properties to show that if A and B are square matrices such that AB is invertible, then both A and B are invertible.
2. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & \alpha & -5 \\ -1 & -1 & 2 \end{bmatrix}$. What values of α makes A *not* invertible?

3. Consider the two subspaces U and W as follows.

$$U = \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} : u_1 - u_2 = 0, u_1 - u_3 = 0 \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$
$$W = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} : w_1 + w_2 + w_3 = 0 \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$



The two subspaces are orthogonal complements (check!). It can be easily seen that the given set of generators bases are actually bases (why?).

(a) Find an orthonormal basis of U. Use it to derive the projection matrix P_U which projects vectors onto the subspace U.

(b) Find an orthonormal basis of W. Use it to derive the projection matrix P_W which projects vectors onto the subspace W.

(c) Notice that $P_U + P_W = I_3$. It turns out that this is not a coincidence. For any orthogonal complement subspaces $U, W \subseteq \mathbb{R}^n$, the sum of their corresponding projection matrices is exactly I_n. Can you explain why it is always the case?
Hint: Use orthogonal decomposition. What exactly is each component of the decomposition?

4. Recall in 2D plane, a mirror matrix M is the matrix for which given a vector x, then Mx is the mirror image of x across a line L that passes through the origin.

(a) What can we tell about M^2 using the following fact: if we reflect a vector v across the line L *twice*, we end up with the original vector v.

(b) Using the answer from part (a), find the possible values of $\det M$.

(c) Take any 2-dimensional region $\mathcal{S}$ you like. What is the region that is produced from left-multiplying any $x \in \mathcal{S}$ by M? In other words, what is the region $\phi_M(\mathcal{S}) = \{Mx : x \in \mathcal{S}\}$?

(d) What is the volume of $\phi_M(\mathcal{S})$? How is to compared to the volume of $\mathcal{S}$? What can we tell about $\det M$?

(e) Recall from problem set 2 that if L makes angle θ with the x-axis, then we have $M = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$. Is the expression consistent with part a and d ?

3 Solutions

1. If AB is invertible, then $\det(AB) \neq 0$. As $\det(AB) = \det A \cdot \det B$, neither $\det A$ not $\det B$ can be zero which means both A and B are invertible.

2. A is not invertible iff $\det A = 0$. There are several ways to compute $\det A$. One approach is to use equation (1): $\det A = (1)(\alpha)(2) + (-2)(-5)(-1) + (1)(3)(-1) - (1)(\alpha)(-1) - (-2)(3)(2) - (1)(-5)(-1) = 3\alpha - 6$.

   Another way to compute the determinant is by deriving ref(A).

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & \alpha & -5 \\ -1 & -1 & 2 \end{bmatrix} \xRightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - 3R_1} \begin{bmatrix} 1 & -2 & 1 \\ 0 & \alpha+6 & -8 \\ 0 & -3 & 3 \end{bmatrix} \xRightarrow{\text{swap } R_2 \& R_3} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & \alpha+6 & -8 \end{bmatrix}$$
$$\xRightarrow{R_3 \leftarrow R_3 + \left(\frac{\alpha+6}{3}\right) R_2} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & \alpha-2 \end{bmatrix} = \text{ref}(A)$$

   We do 1 rows swap and ends up with pivots $1, -3, \alpha-2$. Thus, $\det A = (-1)^1(1)(-3)(\alpha-2) = 3\alpha - 6$.

   Another approach is to perform some row operations then calculate the determinants from block matrices.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & \alpha & -5 \\ -1 & -1 & 2 \end{bmatrix} \xRightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - 3R_1} \begin{bmatrix} 1 & -2 & 1 \\ 0 & \alpha+6 & -8 \\ 0 & -3 & 3 \end{bmatrix}$$

   As we didn't do any swaps, we have

$$\det A = \det\left(\begin{bmatrix} 1 \end{bmatrix}\right) \cdot \det\left(\begin{bmatrix} \alpha+6 & -8 \\ -3 & 3 \end{bmatrix}\right) = 1 \cdot [(\alpha+6)(3) - (-8)(-3)] = 3\alpha - 6.$$

   In any cases, we can solve $\det A = 0$ as $3\alpha - 6 = 0$ which gives us $\alpha = 2$.

3. (a) The basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is already orthogonal since it only has 1 vector. Therefore, we can find the orthonormal basis by normalizing it into $\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$. It follows that

$$P_U = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

   (b) For convenient, let $w_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. The basis $\{w_1, w_2\}$ of W is neither orthogonal nor normalized. In order to obtain orthonormal basis B, we can use Gram-Schmidt. The first vector to be added to B is the normalized

w_1 which is $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = t_1$. The second vector to be added to B is then $t_2 = \text{normalize}(w_2 - \text{proj}_{t_1} w_2)$. We can calculate $w_2 - \text{proj}_{t_1} w_2 = w_2 - (w_2 \cdot t_1)t_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1 \end{bmatrix}$ which gives $t_2 = \text{normalize}\left(\begin{bmatrix} 1/2 \\ -1/2 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$. It follows that

$$\begin{aligned} P_W &= t_1 t_1^\top + t_2 t_2^\top \\ &= \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} + \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix} \\ &= \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}. \end{aligned}$$

(c) Take any vector $v \in \mathbb{R}^n$. From orthogonal decomposition, we can write $v = u + w$ for a unique choice of $u \in U$ and $w \in W$. Moreover, we know that u is the projection of v onto U – so $u = P_U v$. Similarly, we have $w = P_W v$. Thus, we must have $v = u + w = P_U v + P_w v = (P_u + P_w)v$. As the equation holds for arbitrary $v \in \mathbb{R}^n$, we must have $P_u + P_w = I_n$.

4. (a) Pick an arbitrary vector $v \in \mathbb{R}^2$. The first reflection across L is Mv. Then we reflect Mv across L again into $M(Mv) = M^2 v$. However, if we reflect a vector v across L twice, we end up with the original vector v. This means $M^2 v = v$ and it holds for arbitrary $v \in \mathbb{R}^2$. This tells us that $M^2 = I_2$.

(b) Taking determinants of both sides of $M^2 = I_2$, we have $(\det M)^2 = 1$. This implies that $\det M$ is either 1 or -1.

(c) Take any region $\mathcal{S}$ we like. The region $\phi_M(\mathcal{S})$ is a result of applying M to each $x \in \mathcal{S}$ so it becomes $Mx =$ the reflection of x across L. In other words, $\phi_M(\mathcal{S})$ is the reflection of $\mathcal{S}$ across L.

(d) The volume of $\phi_M(\mathcal{S})$ is a *negation* of that of $\mathcal{S}$. This is because reflection across L does not change the area, but flips the handedness of $\mathcal{S}$. As a result, we must have $vol(\phi_M(\mathcal{S})) = -vol(\mathcal{S})$ which means $\det M = -1$.

(e) With $M = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$, we can easily check that

$$M^2 = \begin{bmatrix} \cos^2 2\theta + \sin^2 2\theta & \cos 2\theta \sin 2\theta - \sin 2\theta \cos 2\theta \\ \sin 2\theta \cos 2\theta - \cos 2\theta \sin 2\theta & \sin^2 2\theta + \cos^2 2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

We can also verify that

$$\det M = (\cos 2\theta)(-\cos 2\theta) - (\sin 2\theta)(\sin 2\theta) = -(\cos^2 2\theta + \sin^2 2\theta) = -1.$$

So the properties in part a and d are satisfied.