18.C06 Linear Algebra and Optimization
MIT Fall 2022

Recitation 15

Tuesday Nov 1, 2022

1 TLDR

1.1 Eigenvalues and Eigenvectors

1.1.1 Definition

Let A be a square $n \times n$ matrix. A vector $x \in \mathbb{R}^n$ is called an *eigenvector* iff $Ax = \lambda x$ for some scalar λ. All the scalars λ's satisfying this equation is called an *eigenvalue*.

1.1.2 Characteristic polynomial

Given a square matrix $A \in \mathbb{R}^{n\times n}$. Its characteristic polynomial $p(\cdot)$ is defined as

$$p(\lambda) = \det(\lambda I - A).$$

which is a degree-n polynomial in λ. λ is an eigenvalue of A if and only if it is a root of $p(\lambda)$ – that is $p(\lambda) = 0$.
A matrix $A \in \mathbb{R}^{n\times n}$ can have up to n distinct eigenvalues – as they are roots of a degree-n polynomial $p(\lambda) = 0$.

1.1.3 Properties of Eigenvalues and Eigenvectors

1. The determinant of A is equal to the product of the eigenvalues of A.
2. The trace of A (sum of diagonal elements) is equal to the sum of the eigenvalues of A.
3. If $v_1, ..., v_k$ are eigenvectors associated to distinct eigenvalues $\lambda_1, ..., \lambda_k$, then $v_1, ..., v_k$ are linearly independent.

2 Exercises

1. T/F

   (a) If A has eigenvalue 0, then A is singular.

   (b) If v is an eigenvector of A, then cv where c is a scalar, is also a eigenvector of A.

   (c) If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.

   (d) If (v_1, v_2, v_3) is an eigenvector of A, then (v_1^2, v_2^2, v_3^2) is an eigenvector of A^2.

   (e) If λ is an eigenvalue of A, then λ is also an eigenvalue of A^T.

   (f) If we add 1 to every entry of A, the eigenvalues of A will all increase by 1.

   (g) If we shift A by I, the eigenvalues of A will all shift by 1.

   (h) The real eigenvalues of A^TA must be non-negative.

   (i) If two rows in matrix A are switched, the eigenvalues remain the same.

   (j) If every row of A sum up to k, then k is an eigenvalue of A.

   (k) If every column of A sum up to k, then k is an eigenvalue of A.

2.
$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

   (a) Find the eigenvalues and eigenvectors of A.

   (b) Find the eigenvalues and eigenvectors $2A$.

   (c) Find the eigenvalues and eigenvectors A^2.

   (d) Find the eigenvalues and eigenvectors A^{-1}.

   (e) Find the eigenvalues and eigenvectors $A + 4I$.

3.
$$A = \begin{bmatrix} 4 & 1 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

   (a) Write the characteristic polynomial for A, and find the eigenvalues.

   (b) Find the eigenvector corresponding to each eigenvalue.

4.
$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

   (a) Find the eigenvalues and eigenvectors of A. What would you predict to be the eigenvalues of A^∞?

   (b) $A^2 = \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix}$ Find the eigenvalues and eigenvectors of A^2 using answers from part (a).

   (c) $A^\infty = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$ Find the eigenvalues and eigenvectors of A^∞. Does this match your prediction from part (a)?