# Recitation 15

*Tuesday Nov 1, 2022*

# 1 TLDR

## 1.1 Eigenvalues and Eigenvectors

### 1.1.1 Definition

Let $A$ be a square $n \times n$ matrix. A vector $x \in \mathbb{R}^n$ is called an *eigenvector* iff $Ax = \lambda x$ for some scalar $\lambda$. All the scalars $\lambda$'s satisfying this equation is called an *eigenvalue.*

### 1.1.2 Characteristic polynomial

Given a square matrix $A \in \mathbb{R}^{n \times n}$. Its characteristic polynomial $p(\cdot)$ is defined as

$$p(\lambda) = \det(\lambda I - A).$$

which is a degree-$n$ polynomial in $\lambda$. $\lambda$ is an eigenvalue of $A$ if and only if it is a root of $p(\lambda)$ – that is $p(\lambda) = 0$.
A matrix $A \in \mathbb{R}^{n \times n}$ can have up to $n$ distinct eigenvalues – as they are roots of a degree-$n$ polynomial $p(\lambda) = 0$.

### 1.1.3 Properties of Eigenvalues and Eigenvectors

1. The determinant of $A$ is equal to the product of the eigenvalues of $A$.

2. The trace of $A$ (sum of diagonal elements) is equal to the sum of the eigenvalues of $A$.

3. If $v_1, ..., v_k$ are eigenvectors associated to distinct eigenvalues $\lambda_1, ..., \lambda_k$, then $v_1, ..., v_k$ are linearly independent.

## 2 Exercises

1. T/F
   (a) If $A$ has eigenvalue 0, then $A$ is singular.
   (b) If $v$ is an eigenvector of $A$, then $cv$ where $c$ is a scalar, is also a eigenvector of $A$.
   (c) If $\lambda$ is an eigenvalue of $A$, then $\lambda^2$ is an eigenvalue of $A^2$.
   (d) If $(v_1, v_2, v_3)$ is an eigenvector of $A$, then $(v_1^2, v_2^2, v_3^2)$ is an eigenvector of $A^2$.
   (e) If $\lambda$ is an eigenvalue of $A$, then $\lambda$ is also an eigenvalue of $A^T$.
   (f) If we add 1 to every entry of $A$, the eigenvalues of $A$ will all increase by 1.
   (g) If we shift $A$ by $I$, the eigenvalues of $A$ will all shift by 1.
   (h) The real eigenvalues of $A^TA$ must be non-negative.
   (i) If two rows in matrix $A$ are switched, the eigenvalues remain the same.
   (j) If every row of $A$ sum up to $k$, then $k$ is an eigenvalue of $A$.
   (k) If every column of $A$ sum up to $k$, then $k$ is an eigenvalue of $A$.

2. $$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$
   (a) Find the eigenvalues and eigenvectors of $A$.
   (b) Find the eigenvalues and eigenvectors $2A$.
   (c) Find the eigenvalues and eigenvectors $A^2$.
   (d) Find the eigenvalues and eigenvectors $A^{-1}$.
   (e) Find the eigenvalues and eigenvectors $A + 4I$.

3. $$A = \begin{bmatrix} 4 & 1 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$
   (a) Write the characteristic polynomial for $A$, and find the eigenvalues.
   (b) Find the eigenvector corresponding to each eigenvalue.

4. $$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$
   (a) Find the eigenvalues and eigenvectors of $A$. What would you predict to be the eigenvalues of $A^\infty$?
   (b) $A^2 = \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix}$ Find the eigenvalues and eigenvectors of $A^2$ using answers from part (a).
   (c) $A^\infty = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$ Find the eigenvalues and eigenvectors of $A^\infty$. Does this match your prediction from part (a)?

# 3 Solutions

1. (a) True. $Av = 0v \Rightarrow Av = 0$ has some nontrivial solution, i.e. the nullspace is more than just the zero vector. Therefore $A$ is singular.

   (b) True. $Av = \lambda v \Rightarrow A(cv) = c(Av) = c(\lambda v) = \lambda(cv)$. Therefore $cv$ is also an eigenvector.

   (c) True. $Av = \lambda v \Rightarrow A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda^2 v$. Therefore $\lambda^2$ is also an eigenvalue.

   (d) False. From (c) we see that $v$ is an eigenvector of $A^2$.

   (e) True. The eigenvalues of $A$ are solutions to the equation $det(\lambda I - A) = 0$. Then $det((\lambda I - A)^T) = 0 \Rightarrow det(\lambda I - A^T) = 0$. Therefore $\lambda$ is also an eigenvalue of $A^T$.

   (f) False. Take the matrix $A$ from problem 2 below as an example. You can check that this is not true.

   (g) True. $Av = \lambda v \Rightarrow (A + I)v = \lambda v + v = (\lambda + 1)v$. So $\lambda + 1$ is now an eigenvalue.

   (h) True. Let $v$ be an eigenvector of $A^T A$, i.e. $A^T Av = \lambda v$. Then $\|Av\|^2 = (Av)^T(Av) = v^T A^T Av = v^T \lambda v = \lambda \|v\|^2 \geq 0$. Since $\|Av\|^2$ and $\|v\|^2$ are nonnegative, $\lambda$ must be nonnegative.

   (i) False. Take matrix $A$ from problem 4 below as an example and check that this is not true.

   (j) True. Since every row sums to $k$, let $v = [1\ 1\ ...\ 1]^T$, then $Av$ sums up every entry in each row and is equal to $[k\ k ...\ k]^T$. Therefore, $Av = kv$. $k$ is hence an eigenvalue of $A$.

   (k) True. The same vector $v$ from part (j) is an eigenvector for $A^T$, and $A^T v = kv$. So $k$ is an eigenvalue of $A^T$. Then applying what we have shown in part (e), we see that $k$ is also an eigenvalue of $A$.

2. (a)

$$\begin{aligned}
det(\lambda I - A) &= 0 \\
(2 - \lambda)^2 - (-1)^2 &= 0 \\
\lambda^2 - 4\lambda + 3 &= 0 \\
(\lambda - 1)(\lambda - 3) &= 0 \\
\lambda_1 = 1, \lambda_2 &= 3
\end{aligned}$$

   To find the eigenvectors,

$$\begin{aligned}
Av_1 &= v_1 \\
(A - I)v_1 &= 0 \\
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} v_1 &= 0 \\
v_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}$$

   Similarly, for $v_2$, we have $(A - 3I)v_2 = 0$ and $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b)

$$Av = \lambda v$$
$$2Av = (2\lambda)v$$

The eigenvalues are double the value of those of $A$, i.e. $\lambda = 2, 6$, with eigenvectors $v_1, v_2$ from part (a), respectively.

(c)

$$A^2 v = AAv = A(\lambda v) = \lambda(Av) = \lambda^2 v$$

The eigenvalues are 1 and 9, with eigenvectors $v_1, v_2$ from part (a), respectively.

(d)

$$Av = \lambda v$$
$$A^{-1}\lambda v = v$$
$$A^{-1}v = \frac{1}{\lambda}v$$

The eigenvalues are 1 and 1/3, with eigenvectors $v_1$ and $v_2$ from part (a), respectively.

(e)

$$Av = \lambda v$$
$$(A + 4I)v = Av + 4v = (\lambda + 4)v$$

The eigenvalues are 5 and 7, with eigenvectors $v_1$ and $v_2$ from part (a), respectively.

3. (a)

$$p(\lambda) = det(\lambda I - A) = (\lambda - 4)(\lambda - 2)(\lambda - 9) = 0$$

$\lambda_1 = 4, \lambda_2 = 2, \lambda_3 = 9$.

(b) Solve for $(A - 4I)v_1 = 0, (A - 2I)v_2 = 0, (A - 9I)v_3 = 0$. We get $v_1 = (1, 0, 0), v_2 = (-120), v_3 = (\frac{9}{7}, \frac{3}{7}, 1)$.

4. (a) $\lambda_1 = 1, \lambda_2 = 0.5$. $v_1 = (1, 1), v_2 = (1, -1)$. The eigenvalues of $A^\infty$ will approach 1 and 0. $\lambda_1^\infty \to 1, \lambda_2^\infty \to 0$.

(b) The eigenvalues of $A^2$ will be 1 and 0.25, with eigenvectors $v_1$ and $v_2$ from part (a) respectively.

(c) The eigenvalues are 1 and 0. We know that 1 is an eigenvalue because both columns add up to 1. 0 is an eigenvalue because $A^\infty$ is singular. This matches our prediction in (a).