18.C06 Linear Algebra and Optimization
MIT Fall 2022

Recitation 16

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1 Recap

1.1 Eigendecomposition

Let $A \in \mathbb{R}^{n \times n}$. A is diagonalizable if A can be written as

$$A = TDT^{-1}, \tag{1}$$

where $D = Diagonal(\lambda_1, ..., \lambda_n)$, $T = [v_1 \; ... \; v_n]$, and $v_1, ..., v_n$ are the associated eigenvectors. In order for T to be invertible, $\{v_1, ..., v_n\}$ must be linearly independent.
We know that this is true if the eigenvalues are distinct. However, this is not a necessary condition for A to be diagonalizable.

1.2 Algebraic vs. Geometric Multiplicity

1.2.1 Algebraic Multiplicity

The eigenvalue λ has algebraic multiplicity k if the characteristic polynomial $p(t)$ has a factor $(t - \lambda)^k$.

1.2.2 Geometric Multiplicity

The eigenvalue λ has geometric multiplicity K if $\dim (N(\lambda I - A)) = K$.

1.2.3 Diagonalization

Theorem 1. A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if and only if every eigenvalue λ has the same algebraic and geometric multiplicity.

1.3 Power of Matrices

- For a diagonalizable matrix $A = TDT^{-1}$, we can compute $A^k = TD^kT^{-1}$.
- Suppose A is diagonalizable, with eigenvectors $v_1, ..., v_n$ and eigenvalues $\lambda_1, ..., \lambda_n$. Then $v_1, ..., v_n$ and $\lambda_1^k, ..., \lambda_n^k$ and are the eigenvectors and eigenvalues of A^k.

1.3.1 Polynomial

Define a polynomial in matrix A as:

$$q(A) = q_k A^k + q_{k-1} A^{k-1} + \cdots + q_1 A + q_0 I$$

We can also consider such polynomials with "infinitely many terms". These are called power series.

- For example, $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{1}{2}A^2 + \cdots$ is a power series in A.
- If A is diagonalizable, then $q(A)$ can be computed as $q(A) = TD_qT^{-1}$, where $D_q = diagonal(q(\lambda_1), ..., q(\lambda_n))$.

1.3.2 Characteristic Polynomial

Cayley-Hamilton Theorem: if $p(\lambda) = \det(\lambda I - A)$, then $p(A) = 0$.

1.3.3 Applications - Linear Dynamical System

The recursion $x_{k+1} = Ax_k$ defines a linear dynamical system. Assume that A is diagonalizable $A = TDT^{-1}$. Oftentimes we are interested in its long-term behavior, i.e., what is x_k as $k \to \infty$?

$$x_k \ = A^k x_0 \ = TD^kT^{-1}x_0 \ = T \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix} T^{-1}x_0$$

Notice that D^k is the only factor that has dependency on k; that is the behavior of x_k depends on D^k.

- If $|\lambda_i| > 1$, then $\lambda_i^k \to \infty$.
- If $|\lambda_i| < 1$, then $\lambda_i^k \to 0$.
- If $|\lambda_i| < 1$ for every $i = 1, ..., n$, then $D^k \to 0$ as $k \to \infty$, and hence $x_k \to 0$. Sometimes we say this is a **stable** system.
- If x_0 is an eigenvector of A, then $A^k x_0 = \lambda_0^k x_0$.
- If x_0 is a linear combination of eigenvectors, e.g. $x_0 = c_1v_1 + c_2v_2$, then $A^k x_0 = \lambda_1^k c_1 v_1 + \lambda_2^k c_2 v_2$.

2 Exercises

2.1 Eigenvalues

1. Let $A \in \mathbb{R}^{n\times n}$. Show that if A has only one eigenvalue with algebraic multiplicity n and is diagonalizable, then A is a multiple of the identity matrix.

2. Consider the matrix
$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & 0 & 2 \end{bmatrix}$$
   (a) Find the eigenvalues of A.
   (b) For each eigenvalue, find its algebraic and geometric multiplicity.
   (c) Is A diagonalizable? If so, find its diagonalization.

3. In this problem, we will explore one of many applications of diagonalization – solving a recurrence relation. In particular, suppose that we have a sequence of real numbers $x_0 = x_1 = -1$, and $x_{n+1} = 5x_n - 6x_{n-1}$ for any $n \geq 1$. We want to determine a closed form of x_k for any $k \geq 0$.
   (a) Compute x_2, x_3.
   (b) Let $y_i = \begin{bmatrix} x_{i+1} \\ x_i \end{bmatrix}$. What is y_0?
   (c) The relationship $x_{n+2} = 5x_{n+1} - 6x_n$ implies $y_{n+1} = Ay_n$ for a proper choice of 2×2 matrix A. Find A. In other words, find the 2×2 matrix A such that
$$\begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} \quad \text{for any } n \geq 0.$$
   (d) Given the relationship $y_{i+1} = Ay_i$ for any $i \geq 0$, show that $y_k = A^k y_0$.
   (e) Find the eigenvalues λ_1, λ_2 and and their associated eigenvectors v_1, v_2 of A. Find its diagonalization.
   (f) Compute $y_k = A^k y_0$. What is x_k?
   (g) Compute x_0, x_1, x_2, x_3 by the formula in part g. Do they match the answers from part a?
   (h) Alternatively, write y_0 as a linear combination of v_1 and v_2. How does it aid us in computing $y_k = A^k y_0$?

4. Consider the matrix
$$A = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$
   (a) What is the sum of the eigenvalues of A?
   (b) What is the product of the eigenvalues of A?
   (c) What is the sum and product of the eigenvalues of AA^T?

3 Solutions

1. Suppose λ is the eigenvalue with multiplicity n, and $A = TDT^{-1}$. Since $D = diagonal(\lambda_1, ..., \lambda_n)$, $D = \lambda I$. This means $A = T\lambda IT^{-1} = \lambda I$, and hence A must be diagonal.

2. (a) $p(\lambda) = det(A - \lambda I) = (2 - \lambda)^3 = 0$. Note that A is lower triangular, and $det(A) = det(A^T)$, so the determinant of A is also the product of its diagonal entries. A has one eigenvalue $\lambda = 2$.

   (b) The algebraic multiplicity of $\lambda = 2$ is 3. To find the geometric multiplicity, we find $A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 6 & 0 & 0 \end{bmatrix}$. This matrix has rank 1, which implies that its nullspace has dimension 2. So the geometric multiplicity is 2.

   (c) A is not diagonalizable because algebraic and geometric multiplicities are not equal.

3. (a) $x_2 = 5x_1 - 6x_0 = 5(-1) - 6(-1) = 1, x_3 = 5x_2 - 6x_1 = 5(1) - 6(-1) = 11$.

   (b) $y_0 = \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

   (c) $\begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 5x_{n+1} - 6x_n \\ x_{n+1} \end{bmatrix} = x_{n+1} \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix} + x_n \cdot \begin{bmatrix} -6 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$. This gives $A = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}$.

   (d) It follows that $y_k = Ay_{k-1} = A(Ay_{k-2}) = A^2 y_{k-2} = ... = A^k y_0$.

   (e) An eigenvalue λ follows an equation $\det(\lambda I - A) = 0 \iff \det \begin{bmatrix} \lambda - 5 & 6 \\ -1 & \lambda \end{bmatrix} = 0 \iff \lambda^2 - 5\lambda + 6 = 0 \iff (\lambda - 3)(\lambda - 2) = 0$. Thus, the two eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$. We can determine $v_1 \in N(3I - A)$ to be $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $v_2 \in N(2I - A)$ to be $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

   The diagonalization $A = TDT^{-1}$ is therefore given by $T = [v_1 \; v_2] = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ and $D = diagonal(3, 2) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. In addition, we can compute $T^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$.

   (f) $\begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} = y_k = A^k y_0 = TD^kT^{-1}y_0 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3^{k+1} - 2^{k+2} \\ 3^k - 2^{k+1} \end{bmatrix}$. This gives $x_k = 3^k - 2^{k+1}$.

   (g) Plug in $k = 0, 1, 2, 3$ into part g, we have $x_0 = -1, x_2 = -1, x_2 = 1, x_3 = 11$ which match the answers from part a.

   (h) $y_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = v_1 - 2v_2$. Then we can compute $A^k y_0 = A^k(v_1 - 2v_2) = A^k v_1 - 2A^k v_2 = \lambda_1^k v_1 - 2\lambda_2^k v_2 = 3^k \begin{bmatrix} 3 \\ 1 \end{bmatrix} - 2 \cdot 2^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3^{k+1} - 2^{k+2} \\ 3^k - 2^{k+1} \end{bmatrix}$. This gives $x_k = 3^k - 2^{k+1}$.

4. (a) $0+1+1-1=1$

   (b) -2.
   We want to find $\det(A)$. By applying row operations, we convert A to a diagonal matrix $\begin{bmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ We limit our row operations to adding a multiple of one row to another, so the determinant remains the same. $\det(A) = 2(1)(1/2)(-2) = -2$.

   (c) The diagonal entries of AA^T are the inner products of each row of A with itself. So $tr(AA^T) = 2+2+3+2 = 9$. The product of all eigenvalues is equal to $\det(AA^T) = \det(A)\det(A^T) = (-2)(-2) = 4$.