Recitation 17

Tuesday November 8, 2022

Directions Go over problems from Recitation 16 if any were not covered before moving to this recitation.

1 Recap

1.1 Symmetric Matrices

Symmetric matrices are important to study, often appearing in applications such as the Hessian second derivative and optimization problems. A matrix $A \in \mathbb{R}^{n \times n}$ is symmetric if $A = A^\top$.
With real symmetric matrices, all eigenvalues λ_i are real and eigenvectors can always be chosen to be orthogonal. We can then create the eigendecomposition

$$A = TDT^\top \qquad \text{or equivalently} \qquad A = \sum_{i=1}^{n} \lambda_i v_i v_i^\top$$

where A is a real symmetric matrix, T is the matrix of orthogonal eigenvectors, and D is a diagonal matrix of eigenvalues.

1.2 Quadratic Functions as Matrices

Oftentimes, we want to express quadratic functions as matrices to solve in an optimization problem. Any quadratic function $f(x_1, ..., x_n)$ can be expressed as

$$x^\top Ax + b^\top x$$

where A is a symmetric matrix $\in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ are coefficient matrices, and $x = [x_1 \ ... \ x_n]^\top$ represents a vector of n variables.

2 Exercises

Initially go over exercises from previous recitation not covered

1. True or False

   (a) If a matrix A is symmetric and invertible, so is A^{-1}.

   (b) For any matrix A, the matrix AA^T is symmetric.

2. Write the following equations in matrix-vector form $x^TAx + b^Tx$

   (a) $4x^2 - 6xy + 2y^2 + 7x - 35y$

   (b) $\frac{5}{2}x^2 - 2xy - xz + 2y^2 + 3yz + \frac{5}{2}z^2 + 2x - 35y - 47z$

3. Solve the following linear ODE. Use Julia to calculate any inverses of matrices.

$$\frac{dx(t)}{dt} = -6x(t) + 3y(t)$$

$$\frac{dy(t)}{dt} = 4x(t) + 5y(t)$$

3 Solutions

1. (a) True. We need to check $(A^{-1})^T = A^{-1}$. We know $A = A^T$. Since $AA^{-1} = I_n$, transposing both sides we have $(AA^{-1})^T = I$, so $A^{-1})^T A^T = I_n$. This mean that the inverse of A^T is $(A^{-1})^T$. So $(A^{-1})^T = (A^T)^{-1} = A^{-1}$.

   (b) True. Symmetric means equal to its transpose, and $(AA^T)^T = (A^T)^T A^T = AA^T$.

2. (a) $A = \begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix}, b^T = [2 \ -35 \ -47]$

   (b) $A = \frac{1}{2}\begin{bmatrix} 5 & -2 & -1 \\ -2 & 4 & 3 \\ -1 & 3 & 5 \end{bmatrix}, b^T = [2 \ -35 \ -47]$

3. $\begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. The eigenvalues of the A matrix are 6 and -7 and the eigenvectors are $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ respectively.

   Then, we can say that $\begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{bmatrix} = V \begin{bmatrix} 6 & 0 \\ 0 & -7 \end{bmatrix} V^{-1} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ where $V = \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}$.
   We can then do a change of base to say that

   $$r(t) = r(0)e^{6t}$$

   $$s(t) = s(0)e^{-7t}$$

   , where

   $$\begin{bmatrix} \frac{dr(t)}{dt} \\ \frac{ds(t)}{dt} \end{bmatrix} = V^{-1} \begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{bmatrix} \qquad and \qquad \begin{bmatrix} r(0) \\ s(0) \end{bmatrix} = V^{-1} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$$

   Substituting our change of base back into our original equation, we obtain:

   $$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = V \begin{bmatrix} e^{6t} & 0 \\ 0 & e^{-7t} \end{bmatrix} \begin{bmatrix} r(0) \\ s(0) \end{bmatrix} = \begin{bmatrix} e^{6t}r(0) - 3e^{-7t}s(0) \\ 4e^{6t}r(0) - e^{-7t}s(0) \end{bmatrix}$$