Recitation 20

Tuesday November 22, 2022

1 Recap

1.1 Unconstrained Quadratic Programming

Any quadratic function $f(x_1, ..., x_n)$ can be expressed as $x^\top A x + b^\top x$ where A is a symmetric matrix $\in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ are coefficient matrices, and $x = [x_1 \ ... \ x_n]^\top$ represents a vector of n variables.

Our goal is to find x^* that minimizes the objective function f. In other words, we want to find

$$x^* = \operatorname*{argmin}_{x \in \mathbb{R}^n} x^\top A x + b^\top x$$

We might also want to the minimum value of f attained by x^* which is $f(x^*)$.

1.1.1 Simplifying via Eigendecomposition

Using the fact that A is symmetric, we express $A = UDU^\top$, where U is orthogonal ($UU^\top = I$) and D is diagonal of A's eigenvalues. Now, our minimization problem looks like

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} x^\top A z + b^T x \\
&= \min_{x \in \mathbb{R}^n} (x^\top U) D (U^\top x) + b^\top U (U^\top x) \\
&= \min_{z \in \mathbb{R}^n} z^\top D z + \hat{b}^\top z
\end{aligned}
$$

where $z = U^\top x$ and $\hat{b}^\top = b^\top U$.

The expression is equivalent to

$$\min_{z \in \mathbb{R}^n} \left(\sum_{i=1}^n \lambda_i z_i^2 + \hat{b}_i z_i \right) = \sum_{i=1}^n \min_{z_i} \left(\lambda_i z_i^2 + \hat{b}_i z_i \right)$$

For each term of the summation,

- If $\lambda_i > 0$, then the minimum value is $-\hat{b}_i^2/(4\lambda_i)^2$ attained by $z_i = -\hat{b}_i/(2\lambda_i)$.

- If $\lambda_i = 0$, we can find a minimum only if $\hat{b}_i = 0$, which yields the minimum value 0 attained by any $z_i \in \mathbb{R}$.

- If $\lambda_i < 0$, then we cannot find a minimum since $\lambda_i z_i^2 + \hat{b}_i z_i \to -\infty$ when $|z_i| \to +\infty$.

When A is symmetric:

- if all of A's eigenvalues are strictly positive, we say A is *positive definite (pd)*

- if all of A's eigenvalues are non-negative, we say A is *positive semidefinite (psd)*

Altogether, we can find a solution if matrix A is either 1) pd, or 2) psd with additional conditions that $\hat{b} = 0$ whenever $\lambda_i = 0$.

Finally, if the minimum value (w.r.t z_i's) is attainable by z^*, then we can retrive $x^* = (UU^\top)x^* = U(U^\top x^*) = Uz^*$.

1.2 Equality Constrained QP

We are interested in solving a quadratic program (QP) for which the set of *feasible points* is no longer the whole space $\mathbb{R}^n$, but is restricted to some subspace. In particular, we are interested in constraints of the form $\{x \in \mathbb{R}^n \mid Ax = b\}$ for a given $m \times n$ matrix A and m-dimensional vector b. In particular, we can state our problem as:

$$\min_{x \in \mathbb{R}^n \text{ s.t.} Ax=b} \frac{1}{2}x^\top P x + q^\top x =: f(x) \tag{1}$$

where P is an $n \times n$ symmetric matrix, and q is an n-dimensional vector.

2   Exercises

1. The formal definition of positive semidefinite (psd) matrix is given as follows. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be *positive semidefinite* iff any $x \in \mathbb{R}^n$ satisfies $x^\top Ax \geq 0$.

   Show that if A is symmetric, then A is psd if and only if all of its eigenvalue is non-negative.

   Remarks: a similiar analogy applies for positive definite matrices; however, the condition must have become $x^\top Ax > 0$.

2. True/False: explain or give a counter example. It might be easier to think within the framework of symmetric matrices.

   (a) A non-negative matrix is always positive semidefinite.

   (b) A positive semidefinite matrix is always non-negative.

   (c) A positive definite matrix is always invertible.

3. (a) The sum of two positive semidefinite matrices is also positive semidefinite.

   (b) Show that $A^\top A$ is positive semidefinite for any matrix A of any dimensions.

   (c) Let $A \in \mathbb{R}^{n \times n}$ be a positive semidefinite matrix and $B \in \mathbb{R}^{n \times n}$ be another matrix. Show that $B^\top AB$ is also positive semidefinite.

   Hint: an alternative defitiontion of psd from Question 1 could be helpful.

4.
$$\min_{x,y,z} \frac{5}{2}x^2 - 2xy - xz + 2y^2 + 3yz + \frac{5}{2}z^2 + 2x - 35y - 47z$$

   (a) Write the equation above in matrix-vector form $\mathbf{x}^\top A\mathbf{x} + b^\top \mathbf{x}$.

   (b) Does it have a unique solution? If so, find the solution. If no, explain why not.