Recitation 20

Tuesday November 22, 2022

1 Recap

1.1 Unconstrained Quadratic Programming

Any quadratic function $f(x_1, ..., x_n)$ can be expressed as $x^\top Ax + b^\top x$ where A is a symmetric matrix $\in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ are coefficient matrices, and $x = [x_1 \; ... \; x_n]^\top$ represents a vector of n variables.
Our goal is to find x^* that minimizes the objective function f. In other words, we want to find

$$x^* = \operatorname*{argmin}_{x\in\mathbb{R}^n} x^\top Ax + b^\top x$$

We might also want to the minimum value of f attained by x^* which is $f(x^*)$.

1.1.1 Simplifying via Eigendecomposition

Using the fact that A is symmetric, we express $A = UDU^\top$, where U is orthogonal ($UU^\top = I$) and D is diagonal of A's eigenvalues. Now, our minimization problem looks like

$$\begin{aligned}
&\min_{x\in\mathbb{R}^n} x^\top Az + b^T x \\
&= \min_{x\in\mathbb{R}^n} (x^\top U)D(U^\top x) + b^\top U(U^\top x) \\
&= \min_{z\in\mathbb{R}^n} z^\top Dz + \hat{b}^\top z
\end{aligned}$$

where $z = U^\top x$ and $\hat{b}^\top = b^\top U$.
The expression is equivalent to

$$\min_{z\in\mathbb{R}^n}\left(\sum_{i=1}^n \lambda_i z_i^2 + \hat{b_i} z_i\right) = \sum_{i=1}^n \min_{z_i}\left(\lambda_i z_i^2 + \hat{b_i} z_i\right)$$

For each term of the summation,

- If $\lambda_i > 0$, then the minimum value is $-\hat{b_i}^2/(4\lambda_i)^2$ attained by $z_i = -\hat{b_i}/(2\lambda_i)$.
- If $\lambda_i = 0$, we can find a minimum only if $\hat{b_i} = 0$, which yields the minimum value 0 attained by any $z_i \in \mathbb{R}$.
- If $\lambda_i < 0$, then we cannot find a minimum since $\lambda_i z_i^2 + \hat{b_i} z_i \to -\infty$ when $|z_i| \to +\infty$.

When A is symmetric:

- if all of A's eigenvalues are strictly positive, we say A is *positive definite (pd)*
- if all of A's eigenvalues are non-negative, we say A is *positive semidefinite (psd)*

Altogether, we can find a solution if matrix A is either 1) pd, or 2) psd with additional conditions that $\hat{b} = 0$ whenever $\lambda_i = 0$.

Finally, if the minimum value (w.r.t z_i's) is attainable by z^*, then we can retrive $x^* = (UU^\top)x^* = U(U^\top x^*) = Uz^*$.

1.2 Equality Constrained QP

We are interested in solving a quadratic program (QP) for which the set of *feasible points* is no longer the whole space $\mathbb{R}^n$, but is restricted to some subspace. In particular, we are interested in constraints of the form $\{x \in \mathbb{R}^n \mid Ax = b\}$ for a given $m \times n$ matrix A and m-dimensional vector b. In particular, we can state our problem as:

$$\min_{x \in \mathbb{R}^n \text{ s.t.} Ax = b} \frac{1}{2} x^\top P x + q^\top x =: f(x) \tag{1}$$

where P is an $n \times n$ symmetric matrix, and q is an n-dimensional vector.

2 Exercises

1. The formal definition of positive semidefinite (psd) matrix is given as follows. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be *positive semidefinite* iff any $x \in \mathbb{R}^n$ satisfies $x^\top Ax \geq 0$.

   Show that if A is symmetric, then A is psd if and only if all of its eigenvalue is non-negative.

   Remarks: a similiar analogy applies for positive definite matrices; however, the condition must have become $x^\top Ax > 0$.

2. True/False: explain or give a counter example. It might be easier to think within the framework of symmetric matrices.

   (a) A non-negative matrix is always positive semidefinite.

   (b) A positive semidefinite matrix is always non-negative.

   (c) A positive definite matrix is always invertible.

3. (a) The sum of two positive semidefinite matrices is also positive semidefinite.

   (b) Show that $A^\top A$ is positive semidefinite for any matrix A of any dimensions.

   (c) Let $A \in \mathbb{R}^{n \times n}$ be a positive semidefinite matrix and $B \in \mathbb{R}^{n \times n}$ be another matrix. Show that $B^\top AB$ is also positive semidefinite.

   Hint: an alternative defitiontion of psd from Question 1 could be helpful.

4. $$\min_{x,y,z} \frac{5}{2}x^2 - 2xy - xz + 2y^2 + 3yz + \frac{5}{2}z^2 + 2x - 35y - 47z$$

   (a) Write the equation above in matrix-vector form $\mathbf{x}^\top A\mathbf{x} + b^\top \mathbf{x}$.

   (b) Does it have a unique solution? If so, find the solution. If no, explain why not.

3 Solutions

1. Diagonalize $A = UDU^\top$, where U is orthogonal and D is diagonal of A's eigenvalues which are nonnegative. For any vector $x \in \mathbb{R}^n$, consider $x^\top Ax = (x^\top U)D(U^\top x) = (U^\top x)^\top D(U^\top x) = \sum_{i=1}^n \lambda_i (U^\top x)_i^2$.

   ($\Leftarrow$) For any vector $x \in \mathbb{R}^n$, consider $x^\top Ax = \sum_{i=1}^n \lambda_i (U^\top x)_i^2 \geq 0$ since D is diagonal and its diagonal entries are non-negative; i.e. $\lambda_i \geq 0$ for any i.

   ($\Rightarrow$) Suppose, for contrary, that some of A's eigenvalue is negative. For simplicity, let's assume $\lambda_1 < 0$.Choose $x = U \cdot [1\ 0\ 0\ \cdots\ 0]^\top$. We then have $U^\top x = (U^\top U)[1\,0\,0\ \cdots\ 0]^\top = [1\,0\,0\ \cdots\ 0]^\top$. This implies $x^\top Ax = \sum_{i=1}^n \lambda_i (U^\top x)_i^2 = \lambda_1 < 0$ which contradicts the definition of psd. Thus, all eigenvalues of A must be nonnegative.

2. (a) False. Counter example: $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

   (b) False. Counter example: $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

   (c) True. Recall that we can compute determinant by multiplying eigenvalues with multiplicities. If the matrix is pd, all of its eigenvalues are positive; thus, its determinant = product of eigenvalues is positive; i.e. not zero. This implies that the matrix is invertible.

3. (a) Let's say the two psd matrices are A and B. Take any vector x. Notice that $x^\top (A+B)x = x^\top Ax + x^\top Bx \geq 0 + 0 = 0$. This means $A + B$ is also psd.

   (b) Take any vector x. Notice that $x^\top (A^\top A)x = (x^\top A^\top)(Ax) = (Ax)^\top (Ax) = \|Ax\|^2 \geq 0$. This means $A^\top A$ is always psd.

   (c) Take any vector x. Notice that Bx is also a vector in $\mathbb{R}^n$. Since A is psd, it follows that $(Bx)^\top A(Bx) \geq 0$. Alternatively, we can express $x^\top (B^\top AB)x = (x^\top B^\top)A(Bx) = (Bx)^\top A(Bx) \geq 0$. This means $B^\top AB$ is psd.

4. (a) $A = \frac{1}{2}\begin{bmatrix} 5 & -2 & -1 \\ -2 & 4 & 3 \\ -1 & 3 & 5 \end{bmatrix}, b^T = [2\ -35\ -47]$

   (b) Yes, because A is positive definite. The solution is $(x, y, z) = (3, 5, 7)$.