Linear Algebra and Optimization
MIT Fall 2022

Recitation 20

Tuesday November 28, 2022

1 Recap

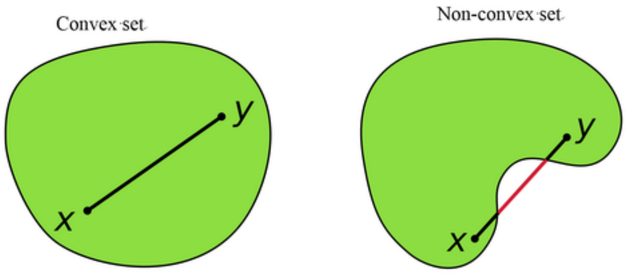
1.1 Convexity

1.1.1 Convex Set

A set C is *convex* if

$$\forall x_1, x_2 \in C, \forall t \in [0,1] : tx_1 + (1-t)x_2 \in C$$

In other words, it contains line segment between any two points in the set.



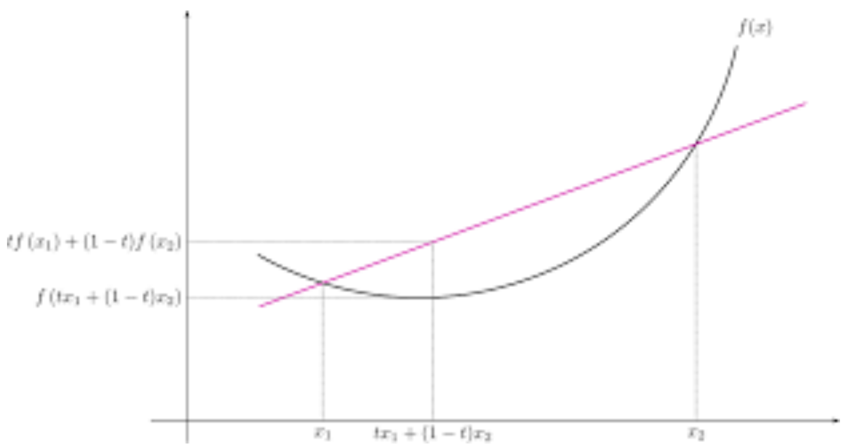
1. If C_1 and C_2 are convex sets, then $C_1 \cap C_2$ is also a convex set.
2. If C is a convex set and ϕ is a linear or affine map, then $\phi(C)$ is also convex.

1.1.2 Convex function

Convex Function: Let $X \subseteq \mathbb{R}^n$ be a convex set. A function $f : X \to \mathbb{R}$ is *convex* if

$$\forall x_1, x_2 \in X, \forall t \in [0,1] : f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2),$$

i.e., the "graph is below the chord".



1. Sum: If $f_1(x)$ and $f_2(x)$ are convex functions, then for $c_1, c_2 \geq 0$, $c_1 f_1(x) + c_2 f_2(x)$ is also a convex function.

2. Pointwise maximum: If $f_1, ..., f_n$ are convex functions, then $f(x) = \max\{f_1(x), ..., f_n(x)\}$ is also convex.

3. Linear change of coordinates: If f is a convex function, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, then $g(x) = f(Ax + b)$ is also convex.

4. Sublevel sets:
$$S_\gamma = \{x \in \mathbb{R}^n : f(x) \leq \gamma\}$$
If $f(x)$ is convex, then S_γ is a convex set.

5. Epigraph of a function:
$$\text{epi}f = \{(x, y) : x \in \mathbb{R}^n, y \in \mathbb{R} : \quad f(x) \leq y\}.$$
If f is a convex function, then epi$f$ is a convex set.

6. Convex Optimization Problem:
$$\begin{aligned} &\min_{x\in\mathbb{R}^n} f(x) \\ &\text{s.t.} \;\; g_i(x) \leq 0 \qquad i = 1, \dots, m \end{aligned}$$
where f and $g_1, \dots, g_m$ are convex functions.

2 Hessian Matrix

- A Hessian matrix of a function f is $H(x) := \nabla^2 f(x) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{ij}$.
- $H(x)$ is always symmetric.
- A function f is convex if and only if $H(x)$ is psd for any x.
- Warning: oftentimes $H(x)$ depends on the values of x. In turns, its eigenvalues depends on the values of x.
- Practice: among the two functions $f(x, y) = 2x^2 - 4xy + y^2$ and $g(x, y) = e^{-2x-y}$, what are their Hessian matrices? Which function has Hessian matrix that depends on x, y and which one does not? Which function is convex?

3 Exercises

1. Suppose U, V are convex sets, and f_1, f_2 are convex functions.
   (a) **(*)** Is $U \cup V$ always a convex set?
   (b) Let $W = \{4u + 2 : \forall u \in U\}$. Is W a convex set?
   (c) Is $f_1 + f_2$ always convex?
   (d) Is $f_1 - f_2$ always convex?
   (e) **(*)** Is $f_1 f_2$ always convex?
   (f) Let $g(x) = \min\{f_1(x), f_2(x)\}$. Is $g(x)$ always a convex function?
   (g) **(*)** Suppose that $f_1 \circ f_2$ is properly defined. Is $f_1 \circ f_2$ always convex? S
2. Let f and g be convex functions and f is non-decreasing; that is $f(x) \leq f(y)$ if $x \leq y$. Show that $f \circ g$ is also convex.
3. Let $C \subseteq \mathbb{R}^n$ be the solution set to a quadratic inequality.
$$C = \{x \in \mathbb{R}^n : x^T Ax + b^T x + c \leq 0\}$$
   with $A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n, c \in \mathbb{R}$. Show that C is a convex set if A is positive semidefinite.
4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and γ be a real number. Are the following sets convex? Give a proof or counterexample (you can assume the sets are nonempty)
   (a) The sub-level set $\{x \in \mathbb{R}^n : f(x) \leq \gamma\}$
   (b) The super-level set $\{x \in \mathbb{R}^n : f(x) \geq \gamma\}$
5. **(*)** Are the following functions convex? Explain your reasoning.
   (a) $f(x, y) = e^{2x-y} + (3x^2 + 2y^2 - xy)$
   (b) $f(x, y, z) = xyz$
   (c) $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \log x_i$