Linear Algebra and Optimization
MIT Fall 2022

Recitation 21

Tuesday November 28, 2022

1 Recap

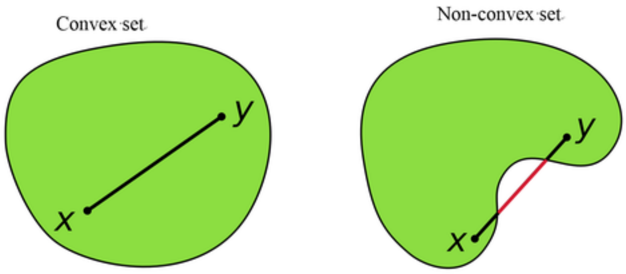
1.1 Convexity

1.1.1 Convex Set

A set C is *convex* if

$$\forall x_1, x_2 \in C, \forall t \in [0,1] : tx_1 + (1-t)x_2 \in C$$

In other words, it contains line segment between any two points in the set.



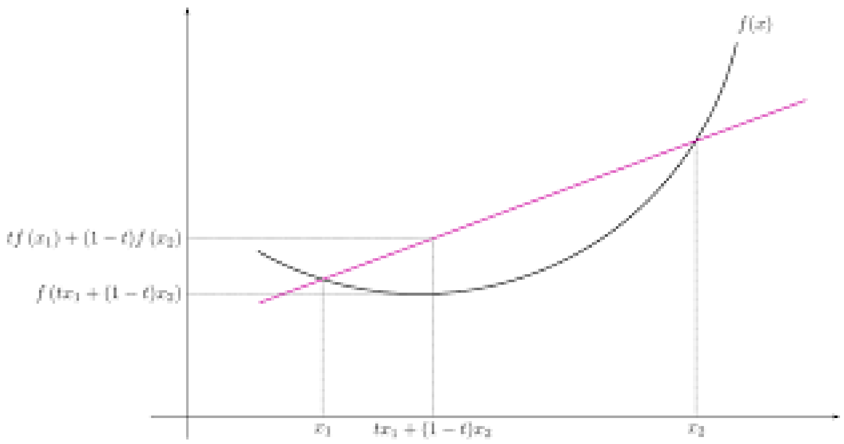
1. If C_1 and C_2 are convex sets, then $C_1 \cap C_2$ is also a convex set.
2. If C is a convex set and ϕ is a linear or affine map, then $\phi(C)$ is also convex.

1.1.2 Convex function

Convex Function: Let $X \subseteq \mathbb{R}^n$ be a convex set. A function $f : X \to \mathbb{R}$ is *convex* if

$$\forall x_1, x_2 \in X, \forall t \in [0,1] : f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2),$$

i.e., the "graph is below the chord".



1. Sum: If $f_1(x)$ and $f_2(x)$ are convex functions, then for $c_1, c_2 \geq 0$, $c_1 f_1(x) + c_2 f_2(x)$ is also a convex function.

2. Pointwise maximum: If $f_1, ..., f_n$ are convex functions, then $f(x) = \max\{f_1(x), ..., f_n(x)\}$ is also convex.

3. Linear change of coordinates: If f is a convex function, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, then $g(x) = f(Ax + b)$ is also convex.

4. Sublevel sets:
$$S_\gamma = \{x \in \mathbb{R}^n : f(x) \leq \gamma\}$$
If $f(x)$ is convex, then S_γ is a convex set.

5. Epigraph of a function:
$$\text{epi}f = \{(x, y) : x \in \mathbb{R}^n, y \in \mathbb{R} : \quad f(x) \leq y\}.$$
If f is a convex function, then epi$f$ is a convex set.

6. Convex Optimization Problem:
$$\begin{aligned} &\min_{x\in\mathbb{R}^n} f(x) \\ &\text{s.t.} \;\; g_i(x) \leq 0 \qquad i = 1, \dots, m \end{aligned}$$
where f and $g_1, \dots, g_m$ are convex functions.

2 Hessian Matrix

- A Hessian matrix of a function f is $H(x) := \nabla^2 f(x) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{ij}$.
- $H(x)$ is always symmetric.
- A function f is convex if and only if $H(x)$ is psd for any x.
- Warning: oftentimes $H(x)$ depends on the values of x. In turns, its eigenvalues depends on the values of x.
- Practice: among the two functions $f(x, y) = 2x^2 - 4xy + y^2$ and $g(x, y) = e^{-2x-y}$, what are their Hessian matrices? Which function has Hessian matrix that depends on x, y and which one does not? Which function is convex?

3 Exercises

1. Suppose U, V are convex sets, and f_1, f_2 are convex functions.
   (a) **(*)** Is $U \cup V$ always a convex set?
   (b) Let $W = \{4u + 2 : \forall u \in U\}$. Is W a convex set?
   (c) Is $f_1 + f_2$ always convex?
   (d) Is $f_1 - f_2$ always convex?
   (e) **(*)** Is $f_1 f_2$ always convex?
   (f) Let $g(x) = \min\{f_1(x), f_2(x)\}$. Is $g(x)$ always a convex function?
   (g) **(*)** Suppose that $f_1 \circ f_2$ is properly defined. Is $f_1 \circ f_2$ always convex? S
2. Let f and g be convex functions and f is non-decreasing; that is $f(x) \leq f(y)$ if $x \leq y$. Show that $f \circ g$ is also convex.
3. Let $C \subseteq \mathbb{R}^n$ be the solution set to a quadratic inequality.
   $$C = \{x \in \mathbb{R}^n : x^T Ax + b^T x + c \leq 0\}$$
   with $A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n, c \in \mathbb{R}$. Show that C is a convex set if A is positive semidefinite.
4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and γ be a real number. Are the following sets convex? Give a proof or counterexample (you can assume the sets are nonempty)
   (a) The sub-level set $\{x \in \mathbb{R}^n : f(x) \leq \gamma\}$
   (b) The super-level set $\{x \in \mathbb{R}^n : f(x) \geq \gamma\}$
5. **(*)** Are the following functions convex? Explain your reasoning.
   (a) $f(x, y) = e^{2x-y} + (3x^2 + 2y^2 - xy)$
   (b) $f(x, y, z) = xyz$
   (c) $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \log x_i$

4 Solutions

1. (a) (This solution is possibly wrong) No. For example, if the two sets are disjoint, then the union does not contain line segment between every two points.

   (b) Yes. $\phi : U \to W$ is an affine transformation.

   (c) Yes. It's a linear combination of convex functions with non-negative coefficients.

   (d) No. Let $f_1(x) = 4x^2, f_2(x) = 6x^2$. Then $f_1(x) - f_2(x) = -2x^2$ is not convex.

   (e) No. Let $f_1(x) = x$ and $f_2(x) = -x$. Then $f_1(x)f_2(x) = -x^2$ is not convex.

   (f) No. Counter example is $f_1 \equiv 0$ and $f_2(x) = x^2 - 1$.

   (g) No. Counter example is $f_1 \equiv x^2$ and $f_2(x) = x^2 - 1$. Then $f_1(f_2(x)) = (x^2-1)^2 = x^4 - 2x^2 + 1$, which is not convex (try to take the second derivative or graph the function).

2. $f_1 \cdot f_2$ is convex because

$$\begin{aligned} f_1 \cdot f_2(tx + (1-t)y) &= f_1(\ f_2(tx + (1-t)y)\ ) \\ &\leq f_1(\ tf_2(x) + (1-t)f_2(y)\ ) \\ &\leq t(f_1 \circ f_2)(x) + (1-t)(f_1 \circ f_2)(y) \end{aligned}$$

   where the first to second line uses the fact that f_2 is convex and f_1 is increasing, and the second to third line uses the fact that f_3 is convex.

3. We want to show that for any $x, y \in$C$, t \in [0,1]$, $tx + (1-t)y$ is in C. Let $f(x) = x^T Ax + b^T x + c$. Then,

$$\begin{aligned} f(tx + (1-t)y) &= (tx + (1-t)y)^T A(tx + (1-t)y) + b^T(tx + (1-t)y) + c \\ &= t^2 x^T Ax + (1-t)^2 y^T Ay + t(1-t)x^T Ay + t(1-t)y^T Ax + tb^T x \\ &+ (1-t)b^T y + tc + (1-t)c \\ &= tx^T Ax + tb^T x + tc + (1-t)y^T Ay + (1-t)b^T y + (1-t)c \\ &+ t(1-t)x^T Ay + t(1-t)y^T Ax - t(1-t)x^T Ax - (1-t)ty^T Ay \\ &\leq t(0) + (1-t)(0) - t(1-t)(x-y)^T A(x-y) \\ &\leq 0 \end{aligned}$$

4. (a) Convex. e^{2x-y} is convex because $f(t) = e^t$ is convex and $(x, y) \to 2x - y$ is a change of coordinates. $3x^2 + 2y^2 - xy$ is also convex because $\begin{bmatrix} 3 & -1/2 \\ -1/2 & 2 \end{bmatrix}$ is psd.

   (b) Not convex. Consider the points $a, b = (1, -1, 1), (-1, 1, 1)$. The we have, $f(\frac{1}{2}a + \frac{1}{2}b) = f(0, 0, 1) = 0 > \frac{1}{2}f(a) + \frac{1}{2}f(b) = -1$. Another proof is to restrict $z = 1$, which is $f(x, y, 1) = xy$. This function is not convex (try to plot it).

   (c) Convex. Each $x_i \log x_i$ term is convex. The sum of convex functions remains convex.