Recitation 25

Tuesday December 13th, 2022

1 Recap - Zero-Sum Games

1.1 Payoff Matrices

Consider a game between two players, Alice and bob, where depending on the strategy each one picks out of a finite set of possibilities, they receive a certain *payoff*. Then, if their payoffs always add up to 0, we say that this game is *zero-sum*, and we can represent the game as a payoff matrix M with the following properties:

- M is in $\mathbb{R}^{m \times n}$, where m is the number of possible strategies for Alice, and n is the number of strategies for Bob.
- $M_{i,j}$ is Alice's payoff when she plays strategy i and Bob plays strategy j. This also implies that Bob's payoff is $-M_{i,j}$

Furthermore, we can encode Alice's strategy as a vector $p \in \mathbb{R}^m$, where p_i is the probability that Alice plays strategy i. Similarly, we can encode Bob's strategy as a vector $q \in \mathbb{R}^n$ where q_j is the probability that Bob plays strategy j. Based on this, we can define a Nash Equilibrium.

1.2 Nash Equilibrium

A Nash Equilibrium is a pair of distributions p and q such that, if Alice's strategy is described by p, then Bob cannot do any better than to choose q and *vice versa*. We are also interested in the *game value*, which is the payoff that the players can guarantee using these strategies. It turns out we can find the Nash Equilibrium using Linear Programming, by setting an order on whether Alice or Bob goes first. If we set Alice to go first, we then want to solve:

$$\begin{gathered}\max \gamma \\ \text{s.t. } p^T A \geq \gamma \vec{1} \\ \vec{0} \leq p \leq \vec{1} \\ (\vec{1})^T p = 1\end{gathered}$$

In words, we are trying to find the maximum γ such that there is a strategy p which guarantees a payoff of at least γ. Now it may seem unfair to make Alice reveal her strategy first, but it turns out that the same game value will be the same if we make Bob go first.

1.3 Duality and Von Neumann's Theorem

Von Neumann's Teorem states that the solution to the above LP is equivalent to the solution for the following LP, which models the game if Bob were forced to reveal his strategy first:

$$\begin{aligned} &\min \lambda \\ \text{s.t. } &\lambda\vec{1} \geq Aq \\ &\vec{0} \leq q \leq \vec{1} \\ &(\vec{1})^T q = 1 \end{aligned}$$

Then, the optimal objective function value for these LPs is precisely equal to the game value.

2 Exercises

1. (Zero-Sum Game) Consider a zero-sum game between Alice and Bob, with the following payoff matrix for Alice:

$$M = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -7 & -3 \\ -2 & 4 & 7 \end{bmatrix}$$

   (a) State the problem of finding a Nash Equilibrium for this game as an LP

   (b) Find a feasible solution for Alice's strategy. What is the expected payoff?

2. A mining company produces 100 tons of red ore and 80 tons of black ore each week. These can be treated in different ways to produce three different alloys, Soft, Hard or Strong. To produce 1 ton of Soft alloy requires 5 tons of red ore and 3 tons of black. For the Hard alloy, the requirements are 3 tons of red and 5 tons of black, whilst for the Strong alloy they are 5 tons of red and 5 tons of black. The profit per ton from selling the alloys are \$250, \$300 and \$400 for Soft, Hard and Strong respectively.

   (a) Formulate the problem of deciding how much of each alloy to make each week as an LP.

3. A hospital wants to make a weekly night shift schedule for nurses. The demand for nurses for the night shift on day j is an integer $d_j, j = 1, ..., 7$. Every nurse works 5 days in a row on the night shift. We want to find the minimal number of nurses the hospital needs to hire.

   (a) What would you define as the decision variables?

   (b) Formulate this problem as an LP.

   (c) Are there additional constraint on the variables other than non-negativity?

3 Solutions

1. (a)

$$\begin{aligned}
&\max \gamma \\
&\text{s.t. } p^T M \geq \gamma \vec{1} \\
&0 \leq p \leq \vec{1} \\
&(\vec{1})^T p = 1
\end{aligned}$$

   (b) A feasible solution for Alice is $p_1 = \frac{6}{13}, p_2 = 0, p_3 = \frac{7}{13}$, which has payoff $\gamma = \frac{43}{13}$. (this is actually the optimal solution)

2. (a) We let x_1, x_2, x_3 be the decision variables of how many tons of soft, hard and strong alloys are produced respectively. Then the LP is as follows:

$$\begin{aligned}
&\max 250x_1 + 300x_2 + 400x_3 \\
&\text{s.t. } 5x_1 + 3x_2 + 5x_3 \leq 100 \\
&3x_1 + 5x_2 + 5x_3 \leq 80 \\
&x_1, x_2, x_3 \geq 0
\end{aligned}$$

3. (a) If we define x_j as the number of nurses that work on day j, we wouldn't be able to express the constraint that every nurse works 5 days in a row. Instead, we could define x_j as the number of nurses starting to work on day j, i.e. x_3 is the number of nurses who work on day 3,4,5,6, and 7.

   (b)

$$\begin{aligned}
&\min_x \sum_{j=1}^{7} x_j \\
&\text{s.t. } x_1 + x_4 + x_5 + x_6 + x_7 \geq d_1 \\
&x_1 + x_2 + x_5 + x_6 + x_7 \geq d_2 \\
&x_1 + x_2 + x_3 + x_6 + x_7 \geq d_3 \\
&x_1 + x_2 + x_3 + x_4 + x_7 \geq d_4 \\
&x_1 + x_2 + x_3 + x_4 + x_5 \geq d_5 \\
&x_2 + x_3 + x_4 + x_5 + x_6 \geq d_6 \\
&x_3 + x_4 + x_5 + x_6 + x_7 \geq d_7 \\
&x_j \geq 0
\end{aligned}$$

   (c) The variables x_j also need to be integers, which is a special constraint for this problem. This is a special class of LP called integer programming.