18.721 PSet 1

Due: Feb 16, 11:59 PM

At the top of your submission, list all the sources you consulted, or write "**Sources consulted:** none" if you did not consult any sources.

1. Let C be a plane curve of degree $d \geq 2$.
   (a) (1 point) Show that for any point $p \in C$, C has multiplicity at most $d-1$ at p.
   (b) (1 point) Show that there is at most one point where C has multiplicity $> \frac{d}{2}$. Conclude that if C is of degree 3, then C has at most one singular point.
2. Consider the set $(\mathbb{P}^2)^4$ of 4-tuples (p, q, r, s) of points in $\mathbb{P}^2$. Call two such 4-tuples projectively equivalent if there is a projective transformation (i.e., change of projective coordinates) sending one to the other.
   (a) (1 point) Classify all equivalence classes under projective equivalence of 4-tuples where the points are NOT all collinear.
   (b) (1 point) When they are all collinear, we can assume without loss of generality that the four points lie in some line $\mathbb{P}^1 \subset \mathbb{P}^2$. In that case, show that each equivalence class contains a unique element of the form $(0, 1, \infty, t)$. (Here, the points in $\mathbb{P}^1$ that we denote by $0, 1$, and ∞ would be written as $(1, 0), (1, 1)$, and $(0, 1)$ in projective coordinates.)
3. Consider the map $f : \mathbb{P}^1 \to \mathbb{P}^3$ sending (x, y) to $(a, b, c, d) = (x^4, x^3y, xy^3, y^4)$.
   (a) (1 point) Show that the image of f is an algebraic variety by exhibiting some homogeneous polynomials which cut out the image. In other words, write down a set of homogeneous polynomials $P_i(a, b, c, d)$ such that $f(\mathbb{P}^1)$ is the loci where all the P_i vanish.
   (b) (1 point) The image of f is a curve in $\mathbb{P}^3$, so one might expect it to be possible to cut it out with only two polynomials. Conjecturally, this is impossible. Check that no two of the polynomials you used in (a) suffice to cut out the image.

4. (2 points) (Exercise 1.11.30, Artin) Find all singular points of the projective plane curve
$$x^3y^2 - x^3z^2 + y^3z^2 = 0$$
and classify them as nodes, cusps, or as other singularities.

5. (2 points) For most of this class, we will be focusing on varieties over $\mathbb{C}$. Some of our theorems do not apply over a field of positive characteristic. As a broad heuristic, anything involving a derivative will work differently in positive characteristic - here is one example.

   Let C be a conic over $\overline{\mathbb{F}}_2$, the algebraic closure of the field with 2 elements. Show that the dual curve of C is a line. In particular, C is not its own bidual.

6. (1 point) Look through the later chapters of Artin's notes (the class text) and find a result or section that you find surprising. Explain what you find surprising about it.