18.721 PSet 1

Due: Feb 16, 11:59 PM

At the top of your submission, list all the sources you consulted, or write "**Sources consulted:** none" if you did not consult any sources.

1. Let C be a plane curve of degree $d \geq 2$.

   (a) (1 point) Show that for any point $p \in C$, C has multiplicity at most $d-1$ at p.

   Solution: Choose a point $q \in C$ other than p. If L is the line through p and q, then C cannot contain L as C is irreducible and not itself a line (as it has degree at least 2). Therefore, C intersects L in d total points (counted with multiplicity.) Since it also intersects L at q, the intersection multiplicity with L at p must be at most $d-1$, so C has multiplicity at most $d-1$ at p.

   (b) (1 point) Show that there is at most one point where C has multiplicity $> \frac{d}{2}$. Conclude that if C is of degree 3, then C has at most one singular point.

   Solution: We use the same idea. If there are two points p, q with multiplicity $> \frac{d}{2}$ then C intersects the line through those two points at more than d points (counted with multiplicity), which gives a contradiction. If C is of degree 3, this tells us C has at most one point with multiplicity > 1, which is equivalent to being singular.

2. Consider the set $(\mathbb{P}^2)^4$ of 4-tuples (p, q, r, s) of points in $\mathbb{P}^2$. Call two such 4-tuples projectively equivalent if there is a projective transformation (i.e., change of projective coordinates) sending one to the other.

   (a) (1 point) Classify all equivalence classes under projective equivalence of 4-tuples where the points are NOT all collinear.

   Solution: We will assume first that the points are distinct (this case is sufficient for a submitted solution to this problem). First assume no three of the points are collinear. Then p, q, and r correspond to three linearly independent vectors in $\mathbb{C}^3$, so there is some linear transformation sending them to $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Let the coordinates of s be (x, y, z). Since no three of the points are collinear,

x, y, and z are all nonzero. Multiplication by the matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

preserves p, q, and r and sends (x, y, z) to (ax, by, cz). So we can choose a, b, and c to send s to $(1, 1, 1)$. In conclusion, all such tuples are equivalent to $((1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1))$.

Now assume that three of the points are collinear. We claim that there is one equivalent class for every possible triple of collinear points, for a total of four equivalent class. WLOG, assume p, q, and t are collinear. As in the previous case, we can take a projective transformation sending p to $(1, 0, 0)$, q to $(0, 1, 0)$, and r to $(0, 0, 1)$. As t is collinear with p and q (but does not coincide with either), it has coordinates $(x, y, 0)$ with x and y nonzero. Again, we can multiply by a matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to get t to have coordinates $(1, 1, 0)$, so our tuple must be equivalent to $((1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0))$.

In the case that the points are not all distinct, there are 6 extra equivalence classes corresponding to which pair of points coincide.

(b) (1 point) When they are all collinear, we can assume without loss of generality that the four points lie in some line $\mathbb{P}^1 \subset \mathbb{P}^2$. In that case, show that each equivalence class contains a unique element of the form $(0, 1, \infty, t)$. (Here, the points in $\mathbb{P}^1$ that we denote by $0, 1$, and ∞ would be written as $(1, 0), (1, 1)$, and $(0, 1)$ in projective coordinates.)

Solution: Any projective transformation of $\mathbb{P}^2$ that sends our line to itself acts on the line by a projective transformation of $\mathbb{P}^1$. We thus can think just about equivalence classes of four points in $\mathbb{P}^1$ under projective transformations. We again assume that all the points are distinct (otherwise, the problem statement is incorrect.)

As p and q are distinct, they correspond to linearly independent vectors in $\mathbb{C}^2$. There is thus some linear transformation sending them to $(1, 0)$ and $(0, 1)$. Say that r has coordinates (x, y). Then the matrix

$$\begin{pmatrix} \frac{1}{x} & 0 \\ 0 & \frac{1}{y} \end{pmatrix}$$

sends r to the point $(1, 1)$. So every equivalence class contains a point of the form $(0, 1, \infty, t)$.

It remains to see that there are no projective transformations sending a point $(0, 1, \infty, u)$ to another point $(0, 1, \infty, v)$ unless $u = v$. Assume our transformation is given by a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

That M sends 0 to 0 implies that $b = 0$, and that M sends ∞ to ∞ implies that $c = 0$. Finally, since M sends 1 to 1, we find that $a = d$, so M is a multiple of the identity and hence induces the trivial projective transformation.

3. Consider the map $f : \mathbb{P}^1 \to \mathbb{P}^3$ sending (x, y) to $(a, b, c, d) = (x^4, x^3y, xy^3, y^4)$.

   (a) (1 point) Show that the image of f is an algebraic variety by exhibiting some homogeneous polynomials which cut out the image. In other words, write down a set of homogeneous polynomials $P_i(a, b, c, d)$ such that $f(\mathbb{P}^1)$ is the loci where all the P_i vanish.

   Solution: We claim that the polynomials $a^3d - b^4$, $ad^3 - c^4$, and $ad - bc$ cut out the image of f. A short calculation shows that these polynomials vanish on the image of f. We need to show that conversely, any point where all three polynomials vanish lies in the image of P.

   Consider a point $(a, b, c, d) \in \mathbb{P}^3$ with $a^3d = b^4, ad^3 = c^4$, and $ad = bc$. If $d = 0$, our equations tell us that $b = c = 0$, so our point must equal $(1, 0, 0, 0)$, which is in the image of f. Otherwise, we can scale the coordinates to assume that $d = 1$. Then we have $a = c^4$. Plugging this into our other equations, we get $c^4 = bc$ and $c^12 = b^4$. If $c = 0$,, then $b^4 = 0$ and $b = 0 = c^3$. Otherwise, we can divide c out from $c^4 = bc$ to also get $b = c^3$. Thus (a, b, c, d) must coincide with the point $f(c, 1)$.

   (b) (1 point) The image of f is a curve in $\mathbb{P}^3$, so one might expect it to be possible to cut it out with only two polynomials. Conjecturally, this is impossible. Check that no two of the polynomials you used in (a) suffice to cut out the image.

   Solution: Many answers are possible, depending on the polynomials chosen in (a). Let us check that every pair of our polynomials have a common root not on $f(\mathbb{P}^1)$.

   The polynomials a^3d-b^4 and ad^3-c^4 have the common zero $(1, 1, -1, 1)$. As ad does not equal bc at this point, it does not lie inside $f(\mathbb{P}^1)$.

   The polynomials a^3d-b^4 and $ad-bc$ have the common zero $(0, 0, 1, 0)$. For this to equal $f(x, y)$, we would need $x = y = 0$, but that would imply $c = 0$, contradiction.

   The polynomials ad^3-c^4 and $ad-bc$ have the common zero $(0, 1, 0, 0)$. For this to equal $f(x, y)$, we would need $x = y = 0$, but that would imply $b = 0$, contradiction.

4. (2 points) (Exercise 1.11.30, Artin) Find all singular points of the projective plane curve
$$x^3y^2 - x^3z^2 + y^3z^2 = 0$$
and classify them as nodes, cusps, or as other singularities.

Solution: The x, y, and z derivatives of our polynomial are $3x^2y^2 - 3x^2z^2$, $2x^3y + 3y^2z^2$, and $-3x^2z^2 + 2y^3z$, respectively. The singular points are the points where all three of these polynomials vanish.

The equation $3x^2y^2 - 3x^2z^2 = 0$ factors as
$$3x^2(y - z)(y + z) = 0,$$
so either $x = 0$, $y = z$, or $y = -z$. If $x = 0$, we can plug that into one of our other equations to see that y or z vanish. This gives us two singular points $(0, 0, 1)$ and $(0, 1, 0)$.

Next, if $y = z$, the vanishing of $2x^3y + 3y^2z^2$ and $-3x^2z^2 + 2y^3z$ become the equations
$$2x^3y + 3y^4 = 0$$
and
$$-3x^2y^2 + 2y^4 = 0.$$
The case where $y = 0$ gives us another singular point $(1, 0, 0)$. If y is not zero, then we must have $2x^3 = -3y^3$ and $2y^2 = 3x^2$, which cannot be satisfied simultaneously. A very similar analysis shows that the case $y = -z$ only gives the singular point $(1, 0, 0)$ that we already found.

It remains to classify the singularity types. Luckily, the simple form of the singular points makes it easy to see the local expansion around them. For instance, around $(0, 0, 1)$, we can dehomogenize by setting $z = 1$ to get the equation
$$(-x^3 + y^3) + x^3y^2 = 0$$
which is already expanded around the origin. The lowest term is cubic, so our curve has multiplicity 3 at $(0, 0, 1)$, and so it has neither a node or a cusp there.

Near $(1, 0, 0)$, we set $x = 1$ to get the equation
$$y^2 - z^2 + y^3z^2 = 0.$$
As the quadratic term $y^2 - z^2$ factors into two linearly independent linear terms, this is a node singularity.

Finally, near $(0, 1, 0)$, we set $y = 1$ and get
$$z^2 + x^3 - x^3z^2 = 0.$$
Here the lowest term is the degenerate quadratic z^2. To see if this gives a cusp, we need to look at the coefficient of x^3. As this coefficient is nonzero, we indeed get a cusp singularity at this point.

5. (2 points) For most of this class, we will be focusing on varieties over $\mathbb{C}$. Some of our theorems do not apply over a field of positive characteristic. As a broad heuristic, anything involving a derivative will work differently in positive characteristic - here is one example.

   Let C be a conic over $\overline{\mathbb{F}}_2$, the algebraic closure of the field with 2 elements. Show that the dual curve of C is a line. In particular, C is not its own bidual.

   Solution: Let P be the polynomial defining C. We can write it explicitly as
$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2.$$

   Because we are in characteristic 2, the x, y, and z derivatives of P are $by + dz, bx + ez$, and $dx + ey$. The dual curve of C is the image of C under the map
$$(x, y, z) \mapsto (by + dz, bx + ez, dx + ey).$$

   Note that
$$(e, d, b) \cdot (by + dz, bx + ez, dx + ey) = 0$$

   so the dual curve will lie in the line corresponding to the point (b, d, e), as desired. Thus C is not its own bidual.

6. (1 point) Look through the later chapters of Artin's notes (the class text) and find a result or section that you find surprising. Explain what you find surprising about it.

   Solution: Up to you.