18.721 PSet 2

Due: Feb 26, 11:59 PM

At the top of your submission, list all the sources you consulted, or write "**Sources consulted:** none" if you did not consult any sources.

1. For the next two problems, we will need to be able to talk about varieties in spaces other than affine or projective space. There is a general notion of an algebraic variety, but for now let us just consider a few straightforward cases.

   (a) (1 point) Consider $\mathbb{A}^n \times \mathbb{P}^m$, with coordinates $x_1, \cdots, x_n$ for $\mathbb{A}^n$ and $y_0, y_1, \cdots, y_m$ for $\mathbb{P}^m$. Call a polynomial $P(x_1, \cdots, x_n, y_0, y_1 \cdots, y_m)$ homogeneous of degree d in the y_i if for any monomial

   $$x_1^{i_1} \cdots x_n^{i_n} y_0^{j_0} y_1^{j_1} \cdots y_m^{j_m}$$

   appearing with a nonzero coefficient in P, we have $j_0 + j_1 + \cdots + j_m = d$. Explain why it makes sense to talk about the locus in $\mathbb{A}^n \times \mathbb{P}^m$ where P vanishes. An algebraic variety in $\mathbb{A}^n \times \mathbb{P}^m$ is defined as the common vanishing locus of some collection of polynomials homogeneous in the y_i.

   (b) (1 point) Now say we want to define a variety in $\mathbb{P}^n \times \mathbb{P}^m$. The key notion for this is that of a bihomogeneous polynomial of bidegree (d, e) in some variables x_i and y_i. This is a polynomial which is homogeneous of degree d when considered as a polynomial in the x_i (with coefficients polynomials of the y_i) and homogeneous of degree e when considered as a polynomial in the y_i (with coefficients polynomials of the x_i).
   Explain why it makes sense to talk about the vanishing locus in $\mathbb{P}^n \times \mathbb{P}^m$ of a bihomogeneous polynomial. A variety in $\mathbb{P}^n \times \mathbb{P}^m$ will be the common vanishing locus of some collection of bihomogeneous polynomials.

   (c) (2 point) We have an embedding (called the Segre embedding) of $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ given by

   $$((a, b), (c, d)) \mapsto (w, x, y, z) = (ac, ad, bc, bd).$$

   It's image is cut out by the single polynomial $wz = xy$ (you can use this fact without proof.) Show that the image of a variety in $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding is a projective variety.

2. In 1.7 of Artin's notes, he introduces the affine blow up of $\mathbb{A}^2$ at one point and uses it to study singularities. Explicitly, his affine blowup is the map $\pi : \mathbb{A}^2 \to \mathbb{A}^2$ defined by
$$(x, w) \mapsto (x, xw).$$
We will introduce the blow-up, which is more commonly used. Let $X \subset \mathbb{A}^2 \times \mathbb{P}^1$ be the set of tuples
$$\{((x, y) \in \mathbb{A}^2, (u, v) \in \mathbb{P}^1) \mid xv = uy\}.$$
This is a variety in the sense of the previous problem. There is a natural map $\pi' : X \to \mathbb{A}^2$ sending a tuple $((x, y), (u, v))$ to (x, y). The space X, with the map π', is known as the blow-up of $\mathbb{A}^2$ at the origin.

   (a) (1 point) Describe the image and fibers of π and π'.

   (b) (1 point) Define an injective map $f : \mathbb{A}^2 \to X$ such that $\pi' \circ f = \pi$. (In fact, f is an open immersion (you do not need to prove this), so this shows that the affine blowup is an open subspace of the full blowup.)

   (c) (2 points) In 1.7.7, Artin shows that for a generic curve C with a node at the origin, there is a curve $C' \subseteq \mathbb{A}^2$ mapped to C by π such that the fiber of $C' \to C$ at the origin is two smooth points. Notably, he only shows this for generic curves (in his notation, he requires a parameter c to be nonzero.) The full blow-up will remove this condition.

   Show that for any curve C with a node at the origin, there is a subvariety $C' \subseteq X$ mapped to C by π' such that the fiber of $C' \to C$ at the origin is two points. (These will turn out to be smooth points of C', but we haven't defined what a smooth point of a general variety is).

3. (2 points, will be easier after Thursday) Consider the product $\mathbb{P}^1 \times \mathbb{P}^1$ of two projective lines. A curve of bidegree (d, e) is a subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$ cut out by a single polynomial bihomogeneous of bidegree (d, e).

   Assume we have two curves C and D which intersect in a finite number of points. Give a formula for the number of intersections (with multiplicity) of C and D, with proof.

   Note: a proof of this formula can be found online with only a moderate amount of effort. While I encourage you to try to solve this problem yourself first (you will learn more), it is allowed to consult online references as long as you write your solution yourself. However, if you do so, you MUST cite your reference, as indicated at the start of this problem set.

4. (1 point) Think about another area of mathematics that you've learned, and tell a story about how an algebraic variety might show up in that area, and what the singularities of that variety might mean. (Your story will not be graded on correctness — the goal is to prime your brain to start thinking about connections with other areas.).