18.721 PSet 2

Due: Feb 26, 11:59 PM

At the top of your submission, list all the sources you consulted, or write "**Sources consulted:** none" if you did not consult any sources.

1. For the next two problems, we will need to be able to talk about varieties in spaces other than affine or projective space. There is a general notion of an algebraic variety, but for now let us just consider a few straightforward cases.

   (a) (1 point) Consider $\mathbb{A}^n \times \mathbb{P}^m$, with coordinates $x_1, \cdots, x_n$ for $\mathbb{A}^n$ and $y_0, y_1, \cdots, y_m$ for $\mathbb{P}^m$. Call a polynomial $P(x_1, \cdots, x_n, y_0, y_1 \cdots, y_m)$ homogeneous of degree d in the y_i if for any monomial

   $$x_1^{i_1} \cdots x_n^{i_n} y_0^{j_0} y_1^{j_1} \cdots y_m^{j_m}$$

   appearing with a nonzero coefficient in P, we have $j_0 + j_1 + \cdots + j_m = d$. Explain why it makes sense to talk about the locus in $\mathbb{A}^n \times \mathbb{P}^m$ where P vanishes. An algebraic variety in $\mathbb{A}^n \times \mathbb{P}^m$ is defined as the common vanishing locus of some collection of polynomials homogeneous in the y_i.

Solution: To check that this locus is well defined in $\mathbb{A}^n \times \mathbb{P}^m$, we need to check that the condition $P(x_1, \cdots, x_n, y_0, \cdots, y_m) = 0$ stays true if we scale all of the y_i by some number c. Our homogeneity condition implies that

$$P(x_1, \cdots, x_n, cy_0, \cdots, cy_m) = c^d P(x_1, \cdots, x_n, y_0, \cdots, y_m) = 0,$$

as desired.

   (b) (1 point) Now say we want to define a variety in $\mathbb{P}^n \times \mathbb{P}^m$. The key notion for this is that of a bihomogeneous polynomial of bidegree (d, e) in some variables x_i and y_i. This is a polynomial which is homogeneous of degree d when considered as a polynomial in the x_i (with coefficients polynomials of the y_i) and homogeneous of degree e when considered as a polynomial in the y_i (with coefficients polynomials of the x_i).

Explain why it makes sense to talk about the vanishing locus in $\mathbb{P}^n \times \mathbb{P}^m$ of a bihomogeneous polynomial. A variety in $\mathbb{P}^n \times \mathbb{P}^m$ will be the common vanishing locus of some collection of bihomogeneous polynomials.

Solution: This is similar to the previous part, but now we need to allow for separate scaling of the x_i and the y_i. The vanishing locus is invariant under such scaling because if $P(x_0, \cdots, x_n, y_0, \cdots, y_m)$ is a bihomogeneous polynomial of bidegree (d,e), then we have

$$P(ax_0, \cdots, ax_n, by_0, \cdots, by_m) = a^d b^e P(x_0, \cdots, x_n, y_0, \cdots, y_m).$$

(c) (2 point) We have an embedding (called the Segre embedding) of $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ given by

$$((a, b), (c, d)) \mapsto (w, x, y, z) = (ac, ad, bc, bd).$$

It's image is cut out by the single polynomial $wz = xy$ (you can use this fact without proof.) Show that the image of a variety in $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding is a projective variety.

Solution: We first do the case where our variety in $\mathbb{P}^1 \times \mathbb{P}^1$ is cut out by a single bihomogeneous polynomial $P(a, b, c, d)$ with bidegree (m, n). Without loss of generality, assume $m \leq n$. Then as a and b cannot both be zero, our variety is also the common zero locus of $a^{n-m}P(a, b, c, d)$ and $b^{n-m}P(a, b, c, d)$, which are both of bidegree (n, n).

We claim that for every bihomogeneous polynomial $Q(a, b, c, d)$ of bidegree (n, n), there is a (necessarily homogeneous) polynomial $R(w, x, y, z)$ with $Q(a, b, c, d) = R(ac, ad, bc, bd)$. It suffices to check this for a monomial $a^i b^j c^k d^l$. The bidegree condition tells us that $i + j = k + l = n$. WLOG, we can assume $i \leq k$. Then, note that

$$a^i b^j c^k d^l = (ac)^i (bc)^{k-i} (bd)^l,$$

so in this case we can take $R(w, x, y, z) = w^i y^{k-i} z^l$.

If $Q(a, b, c, d) = R(ac, ad, bc, bd)$, then the Segre embedding will send the zero locus $V(Q)$ of Q to the common zero locus $V(wz - xy, R)$. In particular, $V(Q)$ is sent to an algebraic variety. Therefore, we see that $V(a^{n-m}P)$ and $V(b^{n-m}P$ are sent to algebraic varieties, so their intersection $V(a^{n-m}P, b^{n-m}P) = V(P)$ is also sent to an algebraic variety, as desired.

In general, if our variety is the common zero locus $V(P_1, \cdots, P_k)$ of multiple polynomials, then it is sent to the intersection of the images of the $V(P_i)$. As the intersection of algebraic varieties is still a variety, this suffices to conclude.

2. In 1.7 of Artin's notes, he introduces the affine blow up of $\mathbb{A}^2$ at one point and uses it to study singularities. Explicitly, his affine blowup is the map $\pi : \mathbb{A}^2 \to \mathbb{A}^2$ defined by
$$(x, w) \mapsto (x, xw).$$
We will introduce the blow-up, which is more commonly used. Let $X \subset \mathbb{A}^2 \times \mathbb{P}^1$ be the set of tuples
$$\{((x, y) \in \mathbb{A}^2, (u, v) \in \mathbb{P}^1) \mid xv = uy\}.$$
This is a variety in the sense of the previous problem. There is a natural map $\pi' : X \to \mathbb{A}^2$ sending a tuple $((x, y), (u, v))$ to (x, y). The space X, with the map π', is known as the blow-up of $\mathbb{A}^2$ at the origin.

(a) (1 point) Describe the image and fibers of π and π'.

Solution: The fibers are described by
$$\pi^{-1}(x, y) = \begin{cases} (x, \frac{y}{x}) & x \neq 0 \\ (1-p)^{n-1}(1-p+4p\alpha\beta)^{n-1} & x = 0, y \neq 0 \\ \emptyset & x = y = 0 \end{cases}$$
and
$$\pi'^{-1}(x, y) = \begin{cases} ((x, y), (x, y)) & (x, y) \neq (0, 0) \\ \mathbb{P}^1 & x = y = 0 \end{cases}.$$
In particular, the image of π is the union of the open set $x \neq 0$ with the origin, while the image of π' is the entire space $\mathbb{A}^2$. This is one indication that π' is a more symmetric object than π.

(b) (1 point) Define an injective map $f : \mathbb{A}^2 \to X$ such that $\pi' \circ f = \pi$. (In fact, f is an open immersion (you do not need to prove this), so this shows that the affine blowup is an open subspace of the full blowup.)

Solution: We can define f by
$$f(x, w) = ((x, xw), (w, 1)).$$
This map is injective because x can be recovered from the tuple $(x, xw) \in \mathbb{A}^2$, and w can be recovered from the tuple $(w, 1) \in \mathbb{P}^1$. It follows immediately from the definition that $\pi' \circ f = \pi$.

(c) (2 points) In 1.7.7, Artin shows that for a generic curve C with a node at the origin, there is a curve $C' \subseteq \mathbb{A}^2$ mapped to C by π such that the fiber of $C' \to C$ at the origin is two smooth points. Notably, he only shows this for generic curves (in his notation, he requires a parameter c to be nonzero.) The full blow-up will remove this condition.

Show that for any curve C with a node at the origin, there is a subvariety $C' \subseteq X$ mapped to C by π' such that the fiber of $C' \to C$ at the origin is two points. (These will turn out to be smooth points of C', but we haven't defined what a smooth point of a general variety is).

Solution: Assume that C is the vanishing locus of $P(x, y)$. Write P as a sum $P_0(x, y) + P_1(x, y) + \cdots + P_d(x, y)$, where $P_i(x, y)$ is homogeneous of degree i. Because we assume C has a node singularity, $P_0 = P_1 = 0$ and $P_2(x, y)$ is the product of two linearly independent linear polynomials $a_1x + b_1y$ and $a_2x + b_2y$.

We will construct a polynomial $Q(u, v, x, y)$ homogeneous of degree 2 in u and v such that

$$u^2P(x, y) \equiv x^2Q(u, v, x, y) \pmod{xv - uy}.$$

It suffices to do the construction when P is a monomial x^ay^b with $a + b \geq 2$. If $a \geq 2$, then we can take $Q(x, y) = u^2x^{a-2}y^b$. On the other hand, if $a < 2$, we can take $Q(x, y) = v^{2-a}y^{a+b-2}$. In both cases, a quick computation verifies the above identity. We can similarly construct $R(u, v, x, y)$ homogeneous of degree 2 in u and v such that

$$v^2P(x, y) \equiv y^2R(u, v, x, y) \pmod{xv - uy}.$$

We define C' to be the common zero locus of Q, R, and $xv - uy$. As $u^2P(x, y)$ and $v^2P(x, y)$ lie in the ideal generated by Q, R, and $xv - uy$, they must also be zero on C'. Since u and v cannot both be zero, this implies that $P(x, y)$ is zero at every point of C', so π' maps C' to C, as desired. (Conversely, it is not hard to show that at a point $(x, y) \neq (0, 0)$ of C, Q and R contain the fiber $\pi'^{-1}(x, y)$.)

It remains to calculate the fiber of C' above $(0, 0)$. Examining the construction of Q and R, we see that only the $P_2(x, y)$ term contributes to $Q(u, v, 0, 0)$ and $R(u, v, 0, 0)$. Then our assumption $P_2(x, y) = (a_1x + b_1y)(a_2x + b_2y)$ gives

$$Q(u, v, 0, 0) = R(u, v, 0, 0) = (a_1u + b_1v)(a_2u + b_2v),$$

so we get two points of C' above $(0, 0)$, corresponding to $(u, v) = (-b_1, a_1)$ and $(u, v) = (-b_2, a_2)$.

3. (2 points, will be easier after Thursday) Consider the product $\mathbb{P}^1 \times \mathbb{P}^1$ of two projective lines. A curve of bidegree (d, e) is a subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$ cut out by a single polynomial bihomogeneous of bidegree (d, e).

   Assume we have two curves C and D which intersect in a finite number of points. Give a formula for the number of intersections (with multiplicity) of C and D, with proof.

Note: a proof of this formula can be found online with only a moderate amount of effort. While I encourage you to try to solve this problem yourself first (you will learn more), it is allowed to consult online references as long as you write your solution yourself. However, if you do so, you MUST cite your reference, as indicated at the start of this problem set.

Solution: Let x and y be coordinates for the first $\mathbb{P}^1$ and let t and z be coordinates for the second $\mathbb{P}^1$. We pick the coordinates so that C and D have no intersections on the line $z = 0$. Assume that C and D have defining polynomials $P(x, y, t, z)$ and $Q(x, y, t, z)$ of bidegrees (c_1, c_2) and (d_1, d_2). Because we assumed that there are no intersections with $z = 0$, we can normalize z to be 1. Write

$$P(x, y, t, 1) = P_0(x, y)t^{c_2} + P_1(x, y)t^{c_2-1} + \cdots + P_{c_2}(x, y)$$

and

$$Q(x, y, t, 1) = Q_0(x, y)t^{d_2} + Q_1(x, y)t^{d_2-1} + \cdots + Q_{d_2}(x, y).$$

By the assumption, the P_i are homogeneous of degree c_1 and the Q_i are homogeneous of degree c_2.

For a given pair (x, y), there is a value of t such that $P(x, y, t, 1) = Q(x, y, t, 1) = 0$ if and only if the resultant $\mathrm{Res}_t(P, Q)$ vanishes at (x, y). (Actually, we haven't carefully justified this in the case where P_0 or Q_0 vanishes — we will discuss this at the end of this solution.)

We claim that the above resultant is a homogeneous polynomial of degree $c_1d_2 + d_1c_2$. It will thus give us $c_1d_2 + d_1c_2$ roots (and hence intersections), counted with multiplicity.

To see this, we recall that the resultant is the determinant of the following $c_2 + d_2 \times c_2 + d_2$ matrix[1]:

$$\begin{pmatrix} P_0 & 0 & \cdots & 0 & Q_0 & 0 & \cdots & 0 \\ P_1 & P_0 & \cdots & 0 & Q_1 & Q_0 & \cdots & 0 \\ P_2 & P_1 & \ddots & 0 & Q_2 & Q_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & P_0 & \vdots & \vdots & \ddots & Q_0 \\ P_{c_2} & P_{c_2-1} & \cdots & \vdots & Q_{d_2} & Q_{d_2-1} & \cdots & \vdots \\ 0 & P_{c_2} & \ddots & \vdots & 0 & Q_{d_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & P_{c_2-1} & \vdots & \vdots & \ddots & Q_{d_2-1} \\ 0 & 0 & \cdots & P_{c_2} & 0 & 0 & \cdots & Q_{d_2} \end{pmatrix}.$$

Every term in the determinant will be the product of entries, one from each column. Each entry in the first d_2 columns is of degree c_1, and every entry in the last c_2 columns is of degree d_2, so every term in the determinant will be of degre $c_1d_2 + d_1c_2$, as desired.

[1] Latex code stolen from https://en.wikipedia.org/wiki/Resultant

Let us now briefly discuss what happens if P_0 or Q_0 vanishes at a point (x, y) (this is not required for a submission to receive full credit.) The argument from class that the resultant is zero if and only if two polynomials have a common root only works if the leading terms of both polynomials are nonzero (this was not an issue in cl ass, because our leading term was just a constant, rather than a function of x and y). However, this statement remains true as long as only one of P_0 and Q_0 vanish - this can be shown by expanding the determinant by minors along the first row.

On the other hand, if both P_0 and Q_0 vanish, then the resultant will be zero, even if $P(x, y, t, 1)$ and $Q(x, y, t, 1)$ do not have a common root. However, this is a feature, not a bug! In this case, C and D will intersect at the line at infinity $z = 0$. We assumed that this does not happen, so P_0 and Q_0 do not have any common roots. (On the other hand, when there is an intersection on the line at infinity, this phenomenon is why the degree of the resultant still gives the correct count.)

4. (1 point) Think about another area of mathematics that you've learned, and tell a story about how an algebraic variety might show up in that area, and what the singularities of that variety might mean. (Your story will not be graded on correctness — the goal is to prime your brain to start thinking about connections with other areas.).

Solution: Many possible answers.