18.721 PSet 3

Due: Mar 1, 11:59 PM

At the top of your submission, list all the sources you consulted, or write "**Sources consulted:** none" if you did not consult any sources.

1. Let $R \subset \mathbb{C}[x, y]$ be the subring of polynomials P such that every monomial $x^i y^j$ appearing with nonzero coefficient in P has even total degree $i + j$.

   (a) (1 point) Prove that R is a finite type $\mathbb{C}$-algebra.

   (b) (1 point) Find, with proof, an embedded affine variety $V \subseteq \mathbb{A}^n$ whose coordinate algebra is isomorphic to R.

2. (2 points) Let A and B be finite type $\mathbb{C}$-algebras. Show that $\operatorname{Spec}(A \oplus B)$ is the disjoint union of $\operatorname{Spec}(A)$ and $\operatorname{Spec} B$ as topological spaces. In particular, $\operatorname{Spec}(A \oplus B)$ is disconnected. (It is conversely true that the spectrum of a ring R is disconnected only if R is a direct sum of two other rings, but you do not need to show this.)

3. One major philosophy of algebraic geometry is that every aspect of the geometry of a variety can be understood via its coordinate algebra. Let's try to understand smoothness of plane curves this way.

   (a) (1 point) Let $P(x, y) \neq 0$ be a polynomial such that $P(0, 0) = 0$, and let C be the vanishing locus of P. Then C is the spectrum of the ring

   $$R = \mathbb{C}[x, y]/(P(x, y)).$$

   Let m be the ideal $(x, y) \subseteq R$. Show that m is a maximal ideal.

   (b) (2 points) Recall that one can multiply ideals, and in particular one can multiply m by itself to get an ideal m^2. Show that C is smooth at the origin if and only if the vector space quotient m/m^2 is 1-dimensional.

4. To study singularities in more depth algebraically, it is helpful to introduce rings of formal power series. The ring $\mathbb{C}[[x_1, \cdots, x_n]]$ is the ring of infinite sums

   $$\sum a_{i_1,\cdots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

   with the natural addition and multiplication operations.

Let C be the vanishing locus of a polynomial $P(x,y) \neq 0$. We will relate how singular C is at $(0,0)$ to the structure of the ring

$$\mathfrak{R} = \mathbb{C}[[x,y]]/(P(x,y)).$$

(This ring is the so-called formal completion of $\mathbb{C}[x,y]/((P(x,y))$ at the origin.)

Let m be the ideal $(x,y) \subseteq \mathbb{C}[[x,y]]$. The ring $\mathbb{C}[[x,y]]$ has a natural metric, where the distance $d(a,b)$ between two elements $a,b \in \mathbb{C}[[x,y]]$ is defined to be 2^{-i}, where i is the largest nonnegative integer with $a-b \in m^i$. (The number 2 is not important here, and can be replaced with any real number > 1.)

(a) (1 point) Show that d defines a metric on $\mathbb{C}[[x,y]]$, and that $\mathbb{C}[[x,y]]$ is complete with respect to this metric.

For parts (b) and (c), assume that C contains and is smooth at the origin. After a change of coordinates, we can assume $P(x,y) = y + S(x,y)$, where $S(x,y)$ only contains terms of total degree at least 2.

(b) (1 point) Let a be an element of $\mathbb{C}[[x,y]]$. Show that for any non-negative integer n, there are elements $b_n \in \mathbb{C}[[x,y]], c_n \in \mathbb{C}[[x]]$, and $d_n \in m^n$ satisfying

$$a = P(x,y)b_n + c_n + d_n.$$

(c) (1 point) Show in fact that there are elements $b \in \mathbb{C}[[x,y]]$ and $c \in \mathbb{C}[[x]]$ satisfying

$$a = P(x,y)b + c.$$

Conclude that the natural map $\mathbb{C}[[x]] \to \mathfrak{R}$ is an isomorphism. In summary, the formal completion of a curve at a smooth point is always isomorphic to $\mathbb{C}[[x]]$, no matter the curve.

(d) (Extra credit, harder, 1 pt) Assume that C has a node at the origin. Show that the ring $\mathfrak{R}$ is isomorphic to $\mathbb{C}[[x,y]]/(xy)$. In particular, it does not depend on C.

This suggests an approach towards classifying singularities: We can say that a singularity of C at p and a singularity of D at q have the same singularity type iff their formal completions are isomorphic.

5. (1 point) Technically, this unit (Chapter 2) does not depend logically on the previous unit (Chapter 1). Imagine that you were teaching a version of this class starting from Chapter 2, instead of Chapter 1. How would that affect how you teach it? Name a specific change you would make.