18.721 PSet 3 Solutions

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1. Let $R \subset \mathbb{C}[x, y]$ be the subring of polynomials P such that every monomial $x^i y^j$ appearing with nonzero coefficient in P has even total degree $i + j$.

   (a) (1 point) Prove that R is a finite type $\mathbb{C}$-algebra.

Solution: We claim it is generated by the monomials x^2, xy, and y^2. Indeed, for a monomial $x^i y^j$ with even total degree, if i and j are even, then we have $x^i y^j = (x^2)^{\frac{i}{2}} (y^2)^{\frac{j}{2}}$. If i and j are odd, then $x^i y^j = xy(x^2)^{\frac{i-1}{2}} (y^2)^{\frac{j-1}{2}}$. Thus, every element of R can be written as a polynomial in terms of x^2, xy, and y^2, so R is generated by those three elements and hence finite type.

   (b) (1 point) Find, with proof, an embedded affine variety $V \subseteq \mathbb{A}^n$ whose coordinate algebra is isomorphic to R.

Solution: There is a map $\mathbb{C}[u, v, w]$ sending u to x^2, v to xy, and w to y^2. By the previous part, this map is surjective, so it corresponds to quotienting by some ideal I of $\mathbb{C}[u, v, w]$. We claim that I is the ideal generated by $uw - v^2$. It follows that I contains $uw - v^2$ from the fact that $(x^2)(y^2) - (xy)^2 = 0$. Let $P(u, v, w)$ be a polynomial with $P(x^2, xy, y^2) = 0$. There is a unique way of writing P as a sum $(uw - v^2)Q(u, v, w) + R(u, v, w)$ such that the degree of v in R is at most 1. Every monomial in u, v, and w with at most one power of v corresponds to a different monomial in x and y, so $R(x^2, xy, y^2)$ cannot be zero unless $R = 0$. Thus, if $P(x^2, xy, y^2) = 0$, then R is the zero polynomial and P is a multiple of $(uw - v^2)$, as desired.

Because R is evidently an integral domain and $\mathbb{C}[u, v, w]/(uw - v^2) \cong R$, we see that $(uw - v^2)$ is a radical ideal. Thus, R is the coordinate algebra of the vanishing locus of $uw - v^2$.

2. (2 points) Let A and B be finite type $\mathbb{C}$-algebras. Show that $\operatorname{Spec}(A \oplus B)$ is the disjoint union of $\operatorname{Spec}(A)$ and $\operatorname{Spec} B$ as topological spaces. In particular, $\operatorname{Spec}(A \oplus B)$ is disconnected. (It is conversely true that the

spectrum of a ring R is disconnected only if R is a direct sum of two other rings, but you do not need to show this.)

Solution: For every maximal ideal m of $\mathrm{Spec}(A)$, we get an ideal $m \oplus B$ of $A \oplus B$. As $(A \oplus B)/(m \oplus B)$ is isomorphic to A/m, which is a field, we see that $m \oplus B$ is a maximal ideal. Simiarly, for any maximal ideal m' of $\mathrm{Spec}(B)$, the ideal $A \oplus m'$ is also maximal. Together, these give us a map of sets $f : \mathrm{Spec}(A) \oplus \mathrm{Spec}(B) \to \mathrm{Spec}(A \oplus B)$.

To show that this map is an isomorphism, we will first prove that every ideal I of $A \oplus B$ is a direct sum of ideals $I_A \oplus I_B$. Indeed, let I_A be the intersection of I with $A \subseteq A \oplus B$ and let I_B be the intersection of I with $B \subseteq A \oplus B$. As I_A and I_B are clearly in I, we see that $I_A \oplus I_B \subseteq I$. Conversely, if (x, y) is an element of I, then multiplying by $(1, 0)$, we see that $(x, 0)$ is an element of I and thus x is an element of I_A. Similarly, y is an element of I_B. Thus, (x, y) is an element of $I_A \oplus I_B$, so I must be equal to $I_A \oplus I_B$.

Assume I is maximal. It is contained in the ideal $A \oplus I_B$, so either we must have $I = A \oplus I_B$ (in which case $I_A = A$) or $A \oplus I_B = A \oplus B$ (in which case $I_B = B$.) If $I_A = A$, then for I to be maximal I_B must be a maximal ideal, and I comes from $\mathrm{Spec}\, B$. Simiarly, if $I_B = B$, then I comes from $\mathrm{Spec}\, A$. This concludes the proof that f is an isomorphism of sets.

It remains to show that the topological structures on both sides agree. The closed sets of $\mathrm{Spec}(A \oplus B)$ are exactly the sets of the form $V(I) = V(I_A \oplus I_B)$. But these are the sets which are the union of sets of the form, $V(I_A)$ and $V(I_B)$, so it coincides with the topology on the disjoint union of $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, B$.

3. One major philosophy of algebraic geometry is that every aspect of the geometry of a variety can be understood via its coordinate algebra. Let's try to understand smoothness of plane curves this way.

   (a) (1 point) Let $P(x, y) \neq 0$ be a polynomial such that $P(0, 0) = 0$, and let C be the vanishing locus of P. Then C is the spectrum of the ring
   $$R = \mathbb{C}[x, y]/(P(x, y)).$$
   Let m be the ideal $(x, y) \subseteq R$. Show that m is a maximal ideal.

Solution: The quotient R/m coincides with the quotient $R/(x, y) \cong \mathbb{C}$, so m is a maximal ideal.

   (b) (2 points) Recall that one can multiply ideals, and in particular one can multiply m by itself to get an ideal m^2. Show that C is smooth at the origin if and only if the vector space quotient m/m^2 is 1-dimensional.

Solution: Let $P(x,y) = P_1(x,y) + P_2(x,y) + \cdots$, where P_i is homogeneous of degree i. Then C is smooth at the origin if and only if $P_1 \neq 0$.

The ideal m^2 in R is the ideal generated by products of two elements in m. As m is generated by x and y, $m^2 = (x^2, xy, y^2)$. Let f be the map $\mathbb{C}[x,y] \to R$. Then $f^{-1}(m)$ is the ideal $(x, y, P(x,y)) = (x,y)$ and $f^{-1}(m^2)$ is the ideal $(x^2, xy, y^2, P(x,y))$. If $P_1 = 0$, then $P(x,y)$ is already in (x^2, xy, y^2) and so $f^{-1}(m^2) = (x^2, xy, y^2)$. Then we have

$$m/m^2 \cong f^{-1}(m)/f^{-1}(m^2) \cong (x,y)/(x^2,xy,y^2) \cong \mathbb{C}x \oplus \mathbb{C}y,$$

and we see that m/m^2 is not 1-dimensional.

Conversely, if P_1 is not zero, we have

$$m/m^2 \cong f^{-1}(m)/f^{-1}(m^2) \cong (x,y)/(x^2,xy,y^2,P(x,y)) \cong \mathbb{C}x \oplus \mathbb{C}y/\mathbb{C}P_1(x,y),$$

and m/m^2 is 1-dimensional.

4. To study singularities in more depth algebraically, it is helpful to introduce rings of formal power series. The ring $\mathbb{C}[[x_1, \cdots, x_n]]$ is the ring of infinite sums

$$\sum a_{i_1,\cdots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

with the natural addition and multiplication operations.

Let C be the vanishing locus of a polynomial $P(x,y) \neq 0$. We will relate how singular C is at $(0,0)$ to the structure of the ring

$$\mathfrak{R} = \mathbb{C}[[x,y]]/(P(x,y)).$$

(This ring is the so-called formal completion of $\mathbb{C}[x,y]/((P(x,y))$ at the origin.)

Let m be the ideal $(x,y) \subseteq \mathbb{C}[[x,y]]$. The ring $\mathbb{C}[[x,y]]$ has a natural metric, where the distance $d(a,b)$ between two elements $a, b \in \mathbb{C}[[x,y]]$ is defined to be 2^{-i}, where i is the largest nonnegative integer with $a - b \in m^i$. (The number 2 is not important here, and can be replaced with any real number > 1.)

(a) (1 point) Show that d defines a metric on $\mathbb{C}[[x,y]]$, and that $\mathbb{C}[[x,y]]$ is complete with respect to this metric.

Solution: Let us check that d satisfies the axioms of a metric. It is clear that d is symmetric. Let us check that the intersection of all the ideals m^i is zero. The ideal m^i contains all formal power series where every term has total degree $\geq i$. As each possible monomial has some finite total degree, it follows that the only element in all the m^i is 0, and thus that $d(a,b) = 0$ if and only if $a = b$.

Now we treat the triangle inequality. Let a, b, and c be distinct elements of $\mathbb{C}[x,y]$. Assume $d(a,b) = 2^{-i}$ and $d(b,c) = 2^{-j}$. Then $a - b \in m^i$ and $b - c \in m^j$, so $a - c = a - b + b - c \in m^{\min(i,j)}$, and $d(a,c) \leq 2^{-\min(i,j)} \leq d(a,b) + d(b,c)$, as desired.

Finally, let us show that $\mathbb{C}[[x,y]]$ is complete. Let a_i be a Cauchy sequence in $\mathbb{C}[[x,y]]$. Then for any integer n, there is some integer N such that for $i, j > N$, we have $d(a_i, a_j) \leq 2^{-n}$, or equivalently $a_i - a_j \in m^n$. Therefore, all the a_i for large enough i are the same mod m^n, or equivalently have the same terms of degree $< i$. Thus, the coefficient $a^i_{c,d}$ of $x^c y^d$ in a_i is the same for all large enough i and must stabilize to some fixed $a_{c,d}$. Then the sequence a_i must stabilize to the formal power series

$$\sum a_{c,d} x^c y^d.$$

For parts (b) and (c), assume that C contains and is smooth at the origin. After a change of coordinates, we can assume $P(x,y) = y + S(x,y)$, where $S(x,y)$ only contains terms of total degree at least 2.

(b) (1 point) Let a be an element of $\mathbb{C}[[x,y]]$. Show that for any non-negative integer n, there are elements $b_n \in \mathbb{C}[[x,y]], c_n \in \mathbb{C}[[x]]$, and $d_n \in m^n$ satisfying

$$a = P(x,y)b_n + c_n + d_n.$$

Solution: We show this by induction. It's true for $n = 0$ by setting $b_0 = c_0 = 0$ and $d_0 = a$. Assume we have an expression

$$a = P(x,y)b_n + c_n + d_n$$

satisfying the conditions. Let the degree n term of d_n be the polynomial $Q(x,y) = sx^n + yR(x,y)$. Then we have

$$\begin{aligned}
a &= P(x,y)b_n + c_n + d_n \\
&= P(x,y)b_n + c_n + (d_n - Q(x,y)) + sx^n + yR(x,y) \\
&= P(x,y)(b_n + R(x,y)) + (c_n + sx^n) + (d_n - Q(x,y) - S(x,y)R(x,y)),
\end{aligned}$$

so we can take $b_{n+1} = b_n + R(x,y)$, $c_{n+1} = c_n + sx^n$, and $d_{n+1} = d_n - Q(x,y) - S(x,y) - R(x,y)$.

(c) (1 point) Show in fact that there are elements $b \in \mathbb{C}[[x,y]]$ and $c \in \mathbb{C}[[x]]$ satisfying

$$a = P(x,y)b + c.$$

Conclude that the natural map $\mathbb{C}[[x]] \to \mathfrak{R}$ is an isomorphism. In summary, the formal completion of a curve at a smooth point is always isomorphic to $\mathbb{C}[[x]]$, no matter the curve.

Solution: We will show that the b_n, c_n, and d_n of the previous problem each form a Cauchy sequence. More precisely, we show that for $i, j > n$, the elements $b_i - b_j, c_i - c_j$, and $d_i - d_j$ lie in m^n. For the d_i this is clear by definitions. For the others, we take the difference of

$$a = P(x, y)b_i + c_i + d_i$$

and

$$a = P(x, y)b_j + c_j + d_j$$

to get

$$0 = P(x, y)(b_i - b_j) + (c_i - c_j) + (d_i - d_j),$$

which implies

$$P(x, y)(b_i - b_j) + (c_i - c_j) \in m^{n+1}.$$

Let $x^r y^s$ be a term in $b_i - b_j$ of minimal total degree. Then there is a term $x^r y^{s+1}$ in $P(x, y)(b_i - b_j)$ of minimal total degree, and it cannot be cancelled out by a term in the $(c_i - c_j)$ (as the c_i are power series in x only.) Therefore, we must have $r+s+1 \geq n+1$, so $b_i - b_j \in m^n$. Plugging this in, we see that $c_i - c_j$ is also in m^n, as desired.

We can now let b and c be the limits of the b_n and the c_n. By definition, the d_n limit to zero, so we get

$$a = P(x, y)b + c.$$

Therefore, every element in $\mathbb{C}[[x, y]]$ is equivalent to an element of $\mathbb{C}[[x]$ mod $P(x, y)$. This implies that the map $\mathbb{C}[[x]] \to \mathfrak{R}$ is surjective. To see injectivity, it suffices to observe that no element of $\mathbb{C}[[x]]$ is a multiple of $P(x, y)$. As y is the lowest degree term of $P(x, y)$, any nonzero multiple of $P(x, y)$ would have to have a term of minimal degree which is a multiple of y, so could not be an element of $\mathbb{C}[[x]]$.

(d) (Extra credit, harder, 1 pt) Assume that C has a node at the origin. Show that the ring $\mathfrak{R}$ is isomorphic to $\mathbb{C}[[x, y]]/(xy)$. In particular, it does not depend on C.

This suggests an approach towards classifying singularities: We can say that a singularity of C at p and a singularity of D at q have the same singularity type iff their formal completions are isomorphic.

Solution: Do a coordinate change so that $P(x, y) = xy + S(x, y)$, with S having only terms of total degree at least 3. The key to this problem is to find $u = x + R_1(x, y)$ and $v = y + R_2(x, y)$ with $P(x, y) = uv$ and the R_i only having terms of total degree at least 2. This can be done with an argument similar to the previous parts, building up the R_i via successive approximation. Once this is done, u and v define a map

$$\mathbb{C}[[u, v]]/(uv) \to \mathfrak{R},$$

and another approximation argument will show that it is an isomorphism.

We note that the hard part of this problem is that there is no a priori natural map of rings $\mathbb{C}[[x,y]]/(xy) \to \mathfrak{R}$, though one can write down a natural map of vector spaces, which is easier to prove is an isomorphism.

5. (1 point) Technically, this unit (Chapter 2) does not depend logically on the previous unit (Chapter 1). Imagine that you were teaching a version of this class starting from Chapter 2, instead of Chapter 1. How would that affect how you teach it? Name a specific change you would make.

Solution: Many possible answers.