# 18.721 PSet 5

Due: Mar 15, 11:59 PM

At the top of your submission, list all the sources you consulted, or write "**Sources consulted:** none" if you did not consult any sources.

1. (1 point, Exercise 3.8.14 in Artin) Describe all morphisms $\mathbb{P}^2 \to \mathbb{P}^1$.

2. In Artin's Section 3.5.26, he considers the projection map

$$\pi : \mathbb{P}^n - \{(0, \cdots, 0, 1)\} \to \mathbb{P}^{n-1}$$

defined by

$$\pi(x_0, x_1, \cdots, x_n) = (x_0, x_1, \cdots, x_n).$$

The fibers of $\pi$ are the lines through $\{0, \cdots, 0, 1\}$ (minus the point itself, which is not in the domain of $\pi$.)

   (a) (1 point) Show that for any point $p \in \mathbb{P}^n$, there is a map $\mathbb{P}^n - \{p\} \to \mathbb{P}^{n-1}$ whose fibers are lines through $p$ (minus $p$ itself).

   (b) (1 point) Show that every plane curve has a non-constant map to $\mathbb{P}^1$.

3. In the previous problem, we showed that every plane curve (in fact, any curve) has a nontrivial to $\mathbb{P}^1$. Conversely, it is very rare that a curve admits a nontrivial map from $\mathbb{P}^1$ - in fact, later in the class we'll see that $\mathbb{P}^1$ does not map to any other smooth curve. For now, let us give an ad hoc argument that the projective cubic plane curve $C$ defined by $x^3 + y^3 + z^3 = 0$ has no nontrivial map from $\mathbb{P}^1$.

   (a) (1 point) Show that every map $\mathbb{P}^1 \to \mathbb{P}^2$ corresponds to a triple (up to scaling) of polynomials $(f(x, y), g(x, y), h(x, y))$, each of which is homogeneous of some degree $d$, such that $f, g$, and $h$ do not have a common root (except for $x = y = 0$.) Conclude that if there is a non-constant map from $\mathbb{P}^1$ to $C$, then there are coprime polynomials $f(t), g(t)$, and $h(t)$ all of degree $d > 0$, with

$$f(t)^3 + g(t)^3 + h(t)^3 = 0.$$

   (b) (1 point) Let the Wronskian of two polynomials $P(t)$ and $Q(t)$ be $W(P, Q) = PQ' - QP'$. (By e.g., $P'$, we mean the derivative of $P$ with respect to $t$.) Show that $W(f(t)^3, g(t)^3) = W(g(t)^3, h(t)^3) = W(h(t)^3, f(t)^3)$. Call this polynomial $w(t)$.

(c) (1 point) Show that $w(t)$ is a multiple of all of $f(t)^2, g(t)^2$, and $h(t)^2$. Derive a contradiction from this fact.

4. (2 points) Show that in a quasi-projective variety, the ring of regular functions on any affine open is of finite type. (Hint: Use the affine communication lemma).

5. For this problem, we'll do some algebraic geometry over $\overline{\mathbb{F}}_p$. You can assume that everything we've proven over the last two weeks applies literally to characteristic $p$ algebraic geometry (which it does). We will also use the version of Bezout's theorem for $\mathbb{P}^1 \times \mathbb{P}^1$ that we proved a few problem sets ago, that the number of intersections (with multiplicity) of a curve of bidegree $(d_1, e_1)$ and a curve of bidegree $(d_2, e_2)$ is $d_1 e_2 + d_2 e_1$.

   (a) (1 point) Let $\mathrm{Fr} : \mathbb{P}^1 \to \mathbb{P}^1$ be the map sending $(x_0, x_1)$ to $(x_0^p, x_1^p)$ (this is called the Frobenius map.) Then we have a diagonal map

   $$\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 : x \mapsto (x, x)$$

   and a twisted diagonal map

   $$\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 : x \mapsto (x, \mathrm{Fr}(x)).$$

   Call the images of these maps $C$ and $C'$. Find the bidegrees of $C$ and $C'$.

   (b) (1 point) Calculate the number of intersections of $C$ and $C'$, and explain why these intersections correspond to points of $\mathbb{P}^1$ with coordinates in $\mathbb{F}_p$.

   For $\mathbb{P}^1$ this is a severely over-complicated way of computing the number of $\mathbb{F}_p$-points, but this is actually the most powerful technique for doing so on a general variety. As one example, it is possible to prove (this is not part of the assignment) that for a smooth projective curve of genus $g$, the number of $\mathbb{F}_p$-points is between $p + 1 - 2g\sqrt{p}$ and $p + 1 + 2g\sqrt{p}$.

6. (1 point) Soon, we'll discuss a classical theorem that a smooth cubic surface has 27 lines on it. On the second floor of the math department, by the main staircase, there is a sculpture exhibit including a cubic surface with 27 lines. Look at this exhibit and convince yourself that the marked lines are indeed lines. Then, look at the nearby exhibits and describe what you learned from them.