18.721 PSet 4

Due: Mar 8, 11:59 PM

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1. Let $R \subset \mathbb{C}[x]$ be the subring of polynomials P such that the coefficient of x in P is zero.

   (a) (1 point) Give an embedding of $\operatorname{Spec} R$ into $\mathbb{A}^2$, and show that the image has a cusp.

Solution: Note that R is generated as a $\mathbb{C}-$algebra by the polynomials x^2 and x^3, which satisfy $(x^3)^2 = (x^2)^3$. There is thus a surjection

$$f : \mathbb{C}[u, v]/(u^3 - v^2) \to R$$

sending u to x^2 and v to x^3. We claim that this map is an isomorphism. Indeed, every element of the left hand side can be uniquely represented as a polynomial in u and v without any terms of degree at least 2 in v. Therefore, as a vector space, $\mathbb{C}[u, v]/(u^3-v^2)$ has a basis given by the u^i and the $u^i v$. As $f(u^i) = x^{2i}$ and $f(u^i v) = x^{2i+3}$, this basis is sent by f to the basis of R consisting of all powers of x except for x itself, which shows that f is an isomorphism.
Now the map

$$\mathbb{C}[u, v] \to \mathbb{C}[u, v]/(u^3 - v^2) \cong R$$

shows that $\operatorname{Spec} R$ is isomorphic to the plane curve $u^3 = v^2$, which has a cusp at the origin.

   (b) (1 point) Find a smooth curve $\operatorname{Spec} S$ with a map $\operatorname{Spec} S \to \operatorname{Spec} R$ which is an isomorphism on topological spaces. Observe that this means that the composition $\operatorname{Spec} S \to \operatorname{Spec} R \to \mathbb{A}^2$ is a closed embedding of topological spaces but not a closed embeddding of algebraic varieties.

Solution: Note that we have a map $R \to \mathbb{C}[x]$, which gives a map of varieties $\mathbb{A}^1 \to \operatorname{Spec} R$. To see that this map is an isomorphism on points, we look at the composition

$$g : \mathbb{A}^1 \to \operatorname{Spec} R \to \mathbb{A}^2$$

defined by

$$x \mapsto (x^2, x^3).$$

We know that $\operatorname{Spec} R$ embeds into $\mathbb{A}^2$ as the vanishing locus of $u^3 - v^2$, so it suffices to show that every (u, v) with $u^3 - v^2 = 0$ can be uniquely expressed as $g(x)$. If $v = 0$, then $u = 0$, and g sends only 0 to $(0, 0)$. On the other hand, if $v \neq 0$, then $x = \frac{u}{v}$ is the unique point sent to (u, v), as desired.

As $\mathbb{A}^1$ and $\operatorname{Spec} R$ both have the cofinite topology (as they are both curves), it follows that the map $\mathbb{A}^1 \to \operatorname{Spec} R$ is also an isomorphism of topological spaces. Thus the map $\mathbb{A}^1 \to \mathbb{A}^2$ is a closed embedding of topological spaces, but is not a closed embedding of varieties because the corresponding map of algebras $\mathbb{C}[u, v] \to R \to \mathbb{C}[x]$ is not surjective, as it factors through R.

2. (2 points) Let R be a finite type $\mathbb{C}$-algebra that is integral (i.e., has no zero-divisors.) Let S be a multiplicative system in R. Show that the localization R_S is a finite type $\mathbb{C}$-algebra if and only if it is isomorphic to the localization R_f at a single nonzero element f. (Recall that R_f is the localization of R at the multiplicative system $\{1, f, f^2, \cdots\}$.

Solution: First we show that R_f is finitely generated. Let R be generated as an algebra by elements $f_1, \cdots, f_n$. Then R_f will be generated by $f_1, \cdots, f_n, \frac{1}{f}$, so is also finite type.

On the other hand, assume R_S is a finite type algebra. Then it is generated by elements $\frac{a_1}{b_1}, \cdots, \frac{a_m}{b_m}$, with $b_i \in S$. Let f be the product of the b_i, which will still be an element of S. Each $\frac{a_i}{b_i}$ can be written as a fraction with denominator f, so all polynomials in those elements can be written as fractions with denominators powers of f. Thus, every element of R_S lies inside R_f, as desired.

3. Our definition of $\operatorname{Spec} R$ as a topological space still makes sense for rings R which are not finite type $\mathbb{C}$-algebras. We will not worry too much about such algebras in this class, but let us briefly discuss the case of $\mathbb{R}$-algebras.

   (a) (1 point) Classify the maximal ideals of $\mathbb{R}[x]$, and describe the map

$$\operatorname{Spec}(\mathbb{C}[x]) \to \operatorname{Spec}(\mathbb{R}[x]).$$

Solution: As $\mathbb{R}[x]$ is a principal ideal domain, the maximal ideals of $\mathbb{R}[x]$ will be those generated by one irreducible polynomial. Thus, we get one maximal ideal $(x - a)$ for every real number a and one maximal ideal $(x^2 + ax + b)$ for every quadratic polynomial with no real roots (equivalently, for every pair of conjugate non-real complex numbers.)

The map $f : \mathbb{C}[x] \to \mathbb{R}[x]$ sends an ideal I to its intersection with $\mathbb{R}[x]$. It is clear that if r is real, f sends $(x - r)$ to $(x - r)$. On the other hand, if r is non-real, then any polynomial with real coefficients and root r must also have $\bar{r}$ as a root and hence be a multiple of $(x-r)(x-\bar{r})$. Thus, f sends $(x-r)$ for non-real r to $((x-r)(x-\bar{r}))$.

(b) (1 point) Classify the maximal ideals of $\mathbb{R}[x,y]/(x^2+y^2+1)$, and describe the map

$$\operatorname{Spec}(\mathbb{C}[x,y]/(x^2+y^2+1)) \to \operatorname{Spec}(\mathbb{R}[x,y]/(x^2+y^2+1)).$$

Note that the vanishing locus of $x^2+y^2+1=0$ in $\mathbb{R}^2$ is empty, and yet we can still study the algebraic geometry of this ring.

Solution: Let g denote the map

$$\operatorname{Spec}(\mathbb{C}[x,y]/(x^2+y^2+1)) \to \operatorname{Spec}(\mathbb{R}[x,y]/(x^2+y^2+1)).$$

We start by claiming that g is surjective. Indeed, let m be a maximal ideal of $\operatorname{Spec}(\mathbb{R}[x,y]/(x^2+y^2+1))$. Then $m \oplus im$ is a non-unit ideal in $\operatorname{Spec}(\mathbb{C}[x,y]/(x^2+y^2+1))$, and is thus contained in some maximal ideal m'. As m' contains (the image of) m, we see that $g(m')$ must be a maximal ideal containing m, hence equaling m.

We know that the maximal ideals of $\operatorname{Spec}(\mathbb{C}[x,y]/(x^2+y^2+1))$ are of the form $((x-a),(y-b))$ for a,b complex numbers with $a^2+b^2+1=0$. Equivalently, we can describe this ideal as containing exactly the polynomials that vanish at (a,b). As a polynomial with real coefficients vanishes at (a,b) if and only if it vanishes at $(\bar{a},\bar{b})$, we see that $g((x-a),(y-b)) = g((x-\bar{a}),(y-\bar{b}))$.

Conversely, we will show that if $g((x-a),(y-b)) = g((x-c),(y-d))$, then either $(c,d)=(a,b)$ or $(c,d)=(\bar{a},\bar{b})$. This will imply that maximal ideals of $\operatorname{Spec}(\mathbb{R}[x,y]/(x^2+y^2+1))$ are classified by pairs (a,b) with $a^2+b^2+1=0$, modulo conjugation. Assume that (c,d) is neither (a,b) nor $(\bar{a},\bar{b})$. Then there are real numbers r and s such that $rc+sd$ is equal to neither $ra+sb$ nor $r\bar{a}+s\bar{b}$. Then the polynomial $(rx+sy-ra-sb)(rx+sy-r\bar{a}-s\bar{b})$ has real coefficients and vanishes at (a,b) but not at (c,d). This gives an element of $g((x-a),(y-b))$ that is not in $g((x-c),(y-d))$.

4. Let S be a subset of $\mathbb{Z}^n$ containing 0 and closed under addition (in other words, a sub-semigroup of $\mathbb{Z}^n$). We can define a ring $\mathbb{C}[S]$ whose elements are formal linear combinations $\sum a_i t^{s_i}$ with the $s_i \in S$, with multiplication determined by the rule $t^{s_i} \cdot t^{s_j} = t^{s_i+s_j}$. An affine toric variety is the spectrum of a ring $\mathbb{C}[S]$. Toric varieties give a large family of easy examples of varieties.

   (a) (1 point) Show that every inclusion $S \subseteq S'$ gives a map of toric varieties $\operatorname{Spec}\mathbb{C}[S'] \to \operatorname{Spec}\mathbb{C}[S]$.

Solution: There is a map of algebras $\mathbb{C}[S] \to \mathbb{C}[S']$ sending t^s to t^s. Taking Spec gives us the desired map.

(b) (1 point) Show that any toric variety has an open subset which is isomorphic to a torus (i.e., the spectrum of an algebra $\mathbb{C}[x_i, x_i^{-1}]$.) This is why these varieties are called toric.

Solution: Let S' be the group generated by S. Then as S is a subgroup of $\mathbb{Z}^n$, it must be isomorphic to $\mathbb{Z}^m$ for some m. It follows that $\mathbb{C}[S']$ is isomorphic to an algebra $\mathbb{C}[x_1, \cdots, x_m, x_1^{-1}, \cdots, x_m^{-1}]$, and hence has spectrum a torus. It remains to show that the map $\operatorname{Spec}\mathbb{C}[S'] \to \operatorname{Spec}\mathbb{C}[S]$ is an open embedding, or equivalently that the map of algebras $\mathbb{C}[S] \to \mathbb{C}[S']$ is a localization. (Technically one needs that it is a localization by one element, but this follows assuming S is finitely generated (a necessary assumption for the problem) by Problem 2.)

Note that the set of elements of the form t^{s_i} is a multiplicative system in $\mathbb{C}[S]$. Inverting these elements gives $\mathbb{C}[S']$, as desired.

5. (2 points) Recall in class that we mentioned that $X = \mathbb{A}^2 - \{(0,0)\}$ is not an affine variety. More precisely, we claim that there is no affine variety Y with a map $\pi : Y \to \mathbb{A}^2$ and two open subvarieties U and V satisfying the following properties:

   - Y is the union of U and V
   - π induces an isomorphism of varieties between U (respectively, V) and the complement of the x-axis (respectively, the y-axis) in $\mathbb{A}^2$
   - π induces an isomorphism of varieties between the intersection $U \cap V$ and the locus where xy does not vanish in $\mathbb{A}^2$.

   Prove this. (Hint: One way of doing this is to think about maps from such a variety Y to $\mathbb{A}^1$.)

Solution: Maps from a variety Y to $\mathbb{A}^1$ are in bijection with elements in $\mathcal{O}(Y)$, so we will work in the language of regular functions. By our assumptions, $\mathcal{O}(U) \cong \mathbb{C}[x, y, y^{-1}]$ and $\mathcal{O}(V) \cong \mathbb{C}[x, y, x^{-1}]$. We also have $\mathcal{O}(U \cap V) \cong \mathbb{C}[x, y, x^{-1}, y^{-1}]$.

Each regular function on Y corresponds to a pair of regular functions, one from each of $\mathcal{O}(U)$ and $\mathcal{O}(V)$, whose restrictions to $\mathcal{O}(U \cap V)$ agree. From our computations above, we see that this implies that $\mathcal{O}(Y) \cong \mathbb{C}[x, y]$. As Y is affine, this implies that $Y \cong \mathbb{A}^2$. But this is a contradiction, as then the origin would be in the image of π.

6. (1 point) Look up the definition of a sheaf. Use google to find as many motivations as you can for why you would define such an object. Elaborate on the one you find most convincing.

Solution: Many possible answers.