18.721 PSet 5

Due: Mar 15, 11:59 PM

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1. (1 point, Exercise 3.8.14 in Artin) Describe all morphisms $\mathbb{P}^2 \to \mathbb{P}^1$.

Solution: Recall that a morphism from a variety X to $\mathbb{P}^1$ corresponds to a pair of rational functions (f, g) on X modulo the equivalence relation $(f, g) \sim (hf, hg)$ for h a nonzero rational function, satisfying the following condition: for any point $x \in X$, there is an open neighborhood U around x and a representative (f, g) of the equivalence class such that f and g are both regular on U and not both zero at x.

A rational function on $\mathbb{P}^2$ is a fraction $\frac{P(x,y,z)}{Q(x,y,z)}$ where P and Q are homogeneous of the same degree. Then for any pair (f, g) of rational functions, we can clear denominators to get a find a pair $(a(x, y, z), b(x, y, z))$ of relatively prime homogeneous polynomials of the same degree such that (a, b) is proportional to (f, g). (We warn the reader that a and b are NOT necessarily rational functions on $\mathbb{P}^2$.) Then any element of the same equivalence class as (f, g) will be expressible as

$$(a(x, y, z) \cdot \frac{c(x, y, z)}{d(x, y, z)}, b(x, y, z) \cdot \frac{c(x, y, z)}{d(x, y, z)}).$$

for c and d homogeneous polynomials with $\deg a + \deg c = \deg d$.

Assume that a and b are not constant. Then by Bezout's theorem, there is a point $p \in X$ with a and b both zero. Let us analyze the behavior of a representative $(a \cdot \frac{c}{d}, b \cdot \frac{c}{d})$ at this point. If d is not zero at p, then both elements of this representative vanish at p. On the other hand, if d vanishes at p, then at least one element of the representative is not regular at p. Either way, the map $\mathbb{P}^2 \to \mathbb{P}^1$ is not defined at p.

In conclusion, the only maps $\mathbb{P}^2 \to \mathbb{P}^1$ are the constant maps.

2. In Artin's Section 3.5.26, he considers the projection map

$$\pi : \mathbb{P}^n - \{(0, \cdots, 0, 1)\} \to \mathbb{P}^{n-1}$$

defined by

$$\pi(x_0, x_1, \cdots, x_n) = (x_0, x_1, \cdots, x_n).$$

The fibers of π are the lines through $\{0, \cdots, 0, 1\}$ (minus the point itself, which is not in the domain of π.)

(a) (1 point) Show that for any point $p \in \mathbb{P}^n$, there is a map $\mathbb{P}^n - \{p\} \to \mathbb{P}^{n-1}$ whose fibers are lines through p (minus p itself).

Solution: Apply a projective transformation to take the point $(0, \cdots, 0, 1)$ to p. The desired map is the composition of this projective transformation with π.

(b) (1 point) Show that every plane curve has a non-constant map to $\mathbb{P}^1$.

Solution: Choose a point p that does not lie in your plane curve C. The restriction of the map from the previous part to C gives a map to $\mathbb{P}^1$. If this map were constant, then C would have to lie in a fiber, and so C would be contained in some line containing p. But this is only possible if C is equal to that line, but we assumed that p is not a point of C.

3. In the previous problem, we showed that every plane curve (in fact, any curve) has a nontrivial to $\mathbb{P}^1$. Conversely, it is very rare that a curve admits a nontrivial map from $\mathbb{P}^1$ - in fact, later in the class we'll see that $\mathbb{P}^1$ does not map to any other smooth curve. For now, let us give an ad hoc argument that the projective cubic plane curve C defined by $x^3 + y^3 + z^3 = 0$ has no nontrivial map from $\mathbb{P}^1$.

   (a) (1 point) Show that every map $\mathbb{P}^1 \to \mathbb{P}^2$ corresponds to a triple (up to scaling) of polynomials $(f(x, y), g(x, y), h(x, y))$, each of which is homogeneous of some degree d, such that f, g, and h do not have a common root (except for $x = y = 0$.) Conclude that if there is a non-constant map from $\mathbb{P}^1$ to C, then there are coprime polynomials $f(t), g(t)$, and $h(t)$ all of degree $d > 0$, with

$$f(t)^3 + g(t)^3 + h(t)^3 = 0.$$

Solution: The the argument we used for Problem 1 also gives the desired classification of maps $\mathbb{P}^1 \to \mathbb{P}^2$. For such a map to land in C, we must have $f(x, y)^3 + g(x, y)^3 + h(x, y)^3$. Without loss of generality, assume that none of f, g, or h vanish at $(1, 0)$, i.e., that they all have a nonzero coefficient of x^d. (We can always apply a projective transformation to $\mathbb{P}^1$ to make this true.) Then taking $f(t) = f(t, 1)$, etc., all the polynomials will still have degree d and be coprime, as desired.

(b) (1 point) Let the Wronskian of two polynomials $P(t)$ and $Q(t)$ be $W(P,Q) = PQ' - QP'$. (By e.g., P', we mean the derivative of P with respect to t.) Show that $W(f(t)^3, g(t)^3) = W(g(t)^3, h(t)^3) = W(h(t)^3, f(t)^3)$. Call this polynomial $w(t)$.

Solution: We check that

$$\begin{aligned} W(f(t)^3, g(t)^3) &= W(f(t)^3, -f(t)^3 - h(t)^3) \\ &= -W(f(t)^3, f(t)^3) - W(f(t)^3, h(t)^3) \\ &= -W(f(t)^3, h(t)^3) \\ &= W(h(t)^3, f(t)^3). \end{aligned}$$

An entirely symmetric argument gives us the other equality.

(c) (1 point) Show that $w(t)$ is a multiple of all of $f(t)^2, g(t)^2$, and $h(t)^2$. Derive a contradiction from this fact.

Solution: We show that $w(t)$ is a multiple of $f(t)^2$ — the other cases will follow by symmetry. Note that

$$w(t) = W(f(t)^3, g(t)^3) = 3f(t)^3 g(t)^2 g'(t) - 3g(t)^3 f(t)^2 f'(t),$$

which is evidently a multiple of $f(t)^2$.

Note that $f(t), g(t)$, and $h(t)$ are pairwise coprime. Indeed, if two of them have a common root, then so does the third, since $f(t)^3 + g(t)^3 + h(t)^3 = 0$. But we assumed that there is no common factor of all three, so they must be pairwise coprime. It follows then that $w(t)$ is a multiple of $f(t)^2 g(t)^2 h(t)^2$. But by the definition of $w(t)$, it is clear that it has degree at most $6d-1$, while $f(t)^2 g(t)^2 h(t)^2$ has degree $6d$, so we must have $w(t) = 0$.

Whenever a Wronskian $W(P,Q)$ vanishes, that means that P and Q are constant multiples of one another. Indeed, if $PQ' - QP' = 0$, then

$$\frac{P}{P'} = \frac{Q}{Q'},$$

which integrates to

$$\log P = \log Q + C,$$

which implies that P and Q are multiples of one another. If we apply this to $w(t)$, we see that $f(t)^3$ and $g(t)^3$ are multiples of one another. But this contradicts coprimality.

This proof is due to Noah Snyder, and proves a more general statement known as the Mason-Stothers theorem. It can be found online in an article entitled "An Alternate Proof of Mason's Theorem".

4. (2 points) Show that in a quasi-projective variety, the ring of regular functions on any affine open is of finite type. (Hint: Use the affine communication lemma).

Solution: Let us show that the hypothesises of the affine communication lemma hold. First, let us show that if we have a ring R and an element $f \in R$, then R_f is finite type if R is. Indeed, R_f is generated by the union of a set of generators of R and $\frac{1}{f}$.

Next, we need to show that if $f_1, \cdots, f_n$ are elements of R with $(f_1, \cdots, f_n)$ the unit ideal, then if all the R_{f_i} are finite type, then so is R. Let R_{f_i} be generated by elements $\frac{a_{ij}}{f_i^{N_i}}$, as i and j range over all possible values. As $(f_1, \cdots, f_n)$ is the unit ideal, we must have some equation $\sum c_i f_i = 1$.

We claim that R is generated by the combination of the a_{ij}, the c_i, and the f_i. Let r be an element of R. The generation statement for the R_{f_i} tells us that for each i, r is equal to a polynomial in the $\frac{a_{ij}}{f_i^{N_i}}$, so there is some integer N such that each rf_i^N is a polynomial in the a_{ij}. We have $(\sum c_i f_i)^{nN} = 1$, which gives us some equation $\sum b_i f_i^N = 1$ where each b_i is equal to a polynomial in the c_i and f_i. This then implies that

$$r = r(\sum b_i f_i^N) = \sum b_i(rf_i^N),$$

and our assumptions imply that all the terms of the RHS can be expressed as polynomials in the a_{ij}, c_i, and f_i, as desired.

To apply the affine communication lemma, note that it suffices to prove the problem statement for projective varieties, as any affine open in a quasi-projective variety will still be an affine open in its closure (in projective space). A projective variety admits a cover by affine subvarieties in $\mathbb{A}^n$, which necessarily have finite type rings of regular functions, so the affine communication lemma applies to tell us that every affine open must have a finite type ring of regular functions.

5. For this problem, we'll do some algebraic geometry over $\overline{\mathbb{F}}_p$. You can assume that everything we've proven over the last two weeks applies literally to characteristic p algebraic geometry (which it does). We will also use the version of Bezout's theorem for $\mathbb{P}^1 \times \mathbb{P}^1$ that we proved a few problem sets ago, that the number of intersections (with multiplicity) of a curve of bidegree (d_1, e_1) and a curve of bidegree (d_2, e_2) is $d_1e_2 + d_2e_1$.

   (a) (1 point) Let $\mathrm{Fr} : \mathbb{P}^1 \to \mathbb{P}^1$ be the map sending (x_0, x_1) to (x_0^p, x_1^p) (this is called the Frobenius map.) Then we have a diagonal map

$$\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 : x \mapsto (x, x)$$

and a twisted diagonal map

$$\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 : x \mapsto (x, \mathrm{Fr}(x)).$$

Call the images of these maps C and C'. Find the bidegrees of C and C'.

Solution: Let coefficients for $\mathbb{P}^1 \times \mathbb{P}^1$ be $((x_0, x_1), (y_0, y_1)$. A defining equation for C is $x_0 y_1 = x_1 y_0$, with bidegree $(1, 1)$. A defining equation for C' is $y_0 x_1^p = y_1 x_0^p$, with bidegree $(p, 1)$.

(b) (1 point) Calculate the number of intersections of C and C', and explain why these intersections correspond to points of $\mathbb{P}^1$ with coordinates in $\mathbb{F}_p$.

Solution: The number of intersections is $p+1$, the same as the number of $\mathbb{F}_p$ points of $\mathbb{P}^1$. Let (a, b) be a point of $\mathbb{P}^1$ with a, b elements in $\mathbb{F}_p$. Then $a^p = a$ and $b^p = b$, so $Fr((a, b)) = (a, b)$. Thus, $((a, b), (a, b))$ would be a point that lies on both C and C'.

For $\mathbb{P}^1$ this is a severely over-complicated way of computing the number of $\mathbb{F}_p$-points, but this is actually the most powerful technique for doing so on a general variety. As one example, it is possible to prove (this is not part of the assignment) that for a smooth projective curve of genus g, the number of $\mathbb{F}_p$-points is between $p + 1 - 2g\sqrt{p}$ and $p + 1 + 2g\sqrt{p}$.

6. (1 point) Soon, we'll discuss a classical theorem that a smooth cubic surface has 27 lines on it. On the second floor of the math department, by the main staircase, there is a sculpture exhibit including a cubic surface with 27 lines. Look at this exhibit and convince yourself that the marked lines are indeed lines. Then, look at the nearby exhibits and describe what you learned from them.

Solution: Many possible answers.