18.721 PSet 6

Due: Mar 22, 11:59 PM

At the top of your submission, list all the sources you consulted, or write "**Sources consulted:** none" if you did not consult any sources.

1. Consider the ring $R \subseteq \mathbb{C}[x, y]$ of polynomials where each monomial $x^i y^j$ with nonzero coefficient has $i + j$ a multiple of n.

   (a) (1 point) Show that R is normal.

   (b) (1 point) Show that the surface $\operatorname{Spec} R$ is not smooth, or equivalently, that there is a maximal ideal m of R such that $\dim m/m^2 > \dim \operatorname{Spec} R = 2$.

2. (2 points) Recall that early on in the class, we stated Hilbert's Nullstellensatz: For an algebraically closed field k, every maximal ideal of $k[x_1, \cdots, x_n]$ is of the form $(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n)$. This was implied by Zariski's lemma, which states that any field extension of k which is finite type as a k-algebra must in fact be isomorphic to k.

   We used a trick to show this for $k = \mathbb{C}$. Use what we've learned about dimension and/or Noether normalization to give another proof of Zariski's lemma. This proof in fact works for any field k.

3. Let X be the projective cubic surface in $\mathbb{P}^3$ defined by the Fermat equation

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0.$$

   (a) (1 point) Write down 27 lines on X explicitly (you do not need to prove that they are all of them, but they are).

   (b) (1 point) Show that each line intersects exactly 10 other lines.

   This suggests that the configuration of lines on this cubic is somewhat symmetric. In fact, the intersection graph of the lines on a smooth cubic surface is always the same, and the symmetry group of this graph has size 51840 and is isomorphic to the symmetry group of the E_6-lattice.

4. (2 points, easier after Tuesday) Now that we have some dimension theory at our disposal, we can rigorously prove some assertions from the first unit. Show that a generic plane curve C of degree $d \geq 2$ has no tritangents, i.e.,

lines that are tangent to C at 3 distinct points. (Hint: Use the method of incidence correspondences. In other words, write down a variety that parametrizes pairs (curve, tritangent) and calculate its dimension.)

5. (2 points, easier after Tuesday, extra credit) Let $\mathrm{Gr}(2,4)$ be the Grassmannian of two-dimensional subspaces in a four dimensional vector space V_4. (Equivalently, it parametrizes lines in $\mathbb{P}^3$, if you would prefer to think in those terms.)

   Choose fixed subspaces V_1, V_2, and V_3 of V_4 such that V_i has dimension i and $V_1 \subset V_2 \subset V_3$. For any nonnegative integers a_1, a_2, and a_3, let $X_{a_1,a_2,a_3} \subset \mathrm{Gr}(2,4)$ be the locus of two-dimensional subspaces W such that for each i, $W \cap V_i$ has dimension a_i. Show that each X_{a_1,a_2,a_3} is either empty or isomorphic to an affine space. (Don't worry too much about rigorously showing the isomorphism at the level of varieties - if you can prove that some algebraic map induces a bijection of sets that's good enough.) Examine the decomposition into affine spaces given by the X_{a_1,a_2,a_3} and explain why it suggests that $\mathrm{Gr}(2,4)$ is not isomorphic to a projective space.

6. (1 point) Choose a theorem from this class whose proof you don't fully understand. Try to explain the proof to somebody else (anybody you want). What theorem did you choose, and what did you learn from the process of explanation?