18.721 PSet 6

Due: Mar 22, 11:59 PM

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1. Consider the ring $R \subseteq \mathbb{C}[x, y]$ of polynomials where each monomial $x^i y^j$ with nonzero coefficient has $i + j$ a multiple of n.

   (a) (1 point) Show that R is normal.

Solution: Let a be an element of the field of fractions $K(R)$ of R. If a is integral over R, then it is the root of a polynomial

$$a^n + r_1 a^{n-1} + \cdots + r_n = 0$$

with the coefficients r_i elements of R. The r_i are thus also elements of $\mathbb{C}[x, y]$, thus a is integral over $\mathbb{C}[x, y]$. As $\mathbb{C}[x, y]$ is normal, this implies that the image of a in $K(\mathbb{C}[x, y])$ in fact lies in $\mathbb{C}[x, y]$. So a is equal to a polynomial $P(x, y)$. It remains to show that if a polynomial $P(x, y)$ is equal to the quotient of two elements of R, then $P(x, y)$ must itself be an element of R. This can be shown in many ways; one way is to consider the map sending (x, y) to $(\zeta x, \zeta y)$, where ζ is a primitive nth root of unity. The elements of R are precisely the polynomials which are invariant under this map, and a quotient of such polynomials is still invariant, which proves the desired claim.

   (b) (1 point) Show that the surface $\operatorname{Spec} R$ is not smooth, or equivalently, that there is a maximal ideal m of R such that $\dim m/m^2 > \dim \operatorname{Spec} R = 2$.

Solution: Let m be the preimage in R of the ideal (x, y) in $\mathbb{C}[x, y]$. Then m is the ideal of elements in R with no constant term, and R/m is isomorphic to $\mathbb{C}$, so m is maximal. The ideal m only contains polynomials of degree at least n, so m^2 only contains polynomials of degree at least $2n$. Thus, $x^n, x^{n-1}y, \cdots, y^n$ give rise to $n + 1$ linearly independent elements (they in fact form a basis) of m/m^2, so $\dim m/m^2 \geq n + 1 > 2$, as long as $n > 1$.

2. (2 points) Recall that early on in the class, we stated Hilbert's Nullstellensatz: For an algebraically closed field k, every maximal ideal of $k[x_1, \cdots, x_n]$ is of the form $(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n)$. This was implied by Zariski's lemma, which states that any field extension of k which is finite type as a k-algebra must in fact be isomorphic to k.

   We used a trick to show this for $k = \mathbb{C}$. Use what we've learned about dimension and/or Noether normalization to give another proof of Zariski's lemma. This proof in fact works for any field k.

Solution: Let K be a field extension of k which is finite type as a k-algebra. As K is a field, the zero ideal is the only maximal ideal, so $\operatorname{Spec} K$ is a single point. Thus, the c-dimension of $\operatorname{Spec} K$ is equal to 0. We showed in class that c-dimension is equal to t-dimension, so the transcendence degree of K over k must be zero as well. Thus, every element of K is algebraic over k, but k is algebraically closed, so K must be isomorphic to k.

3. Let X be the projective cubic surface in $\mathbb{P}^3$ defined by the Fermat equation

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0.$$

   (a) (1 point) Write down 27 lines on X explicitly (you do not need to prove that they are all of them, but they are).

Solution: For any third roots of unity ζ_1, ζ_2, the line parametrized by

$$(\zeta_1 a, -a, \zeta_2 b, -b)$$

gives a line on the Fermat cubic. This gives 9 lines already. Permuting x_0 and x_2 gives another 9, and permuting x_0 and x_3 gives a third set of 9 lines, for a total of 27.

   (b) (1 point) Show that each line intersects exactly 10 other lines.

   This suggests that the configuration of lines on this cubic is somewhat symmetric. In fact, the intersection graph of the lines on a smooth cubic surface is always the same, and the symmetry group of this graph has size 51840 and is isomorphic to the symmetry group of the E_6-lattice.

Solution: All lines here are symmetric, as any line can be sent to any one by some combination of permutation of the x_i and multiplying some x_i by third roots of unity. So it suffices to show that the line

$$(a, -a, b, -b)$$

intersects exactly 10 other lines. It intersects another line of the form

$$(\zeta_1 a, -a, \zeta_2 b, -b)$$

if and only one of the ζ_i is zero, which gives 4 other lines of this type. It intersects a line of the form

$$(\zeta_2 b, -a, \zeta_1 a, -b)$$

if and only if $\zeta_1\zeta_2 = 1$, which gives 3 lines of this type. And it intersects a line of the form

$$(-b, -a, \zeta_2 b, \zeta_1 a)$$

if and only if $\zeta_1 = \zeta_2$, which gives 3 lines of this type, for a total of $4 + 3 + 3 = 10$ lines.

4. (2 points, easier after Tuesday) Now that we have some dimension theory at our disposal, we can rigorously prove some assertions from the first unit. Show that a generic plane curve C of degree $d \geq 2$ has no tritangents, i.e., lines that are tangent to C at 3 distinct points. (Hint: Use the method of incidence correspondences. In other words, write down a variety that parametrizes pairs (curve, tritangent) and calculate its dimension.)

Solution: Let V be the vector space of degree d homogeneous polynomials in 3 variables. Then a plane curve is defined by a point in $\mathbb{P}(V)$. Similarly, a line in $\mathbb{P}^2$ corresponds to a point in the dual plane $\mathbb{P}^{2,\vee}$. let Z be the locus of points (C, l) in $\mathbb{P}(V) \times \mathbb{P}^{2,\vee}$ where l is tritangent to C. (Technically we haven't shown that this a variety, which is straightforward but tedious.)

Let us compute the dimension of Z. The fiber of Z over a point l in $\mathbb{P}^{2,\vee}$ is the subset $X \subseteq \mathbb{P}(V)$ of curves tritangent to l. In the language of polynomials, it corresponds to the sublocus of V of polynomials whose restriction to l have at least three multiple zeroes. Note that choosing a nonzero polynomial function on l amounts to choosing n roots (with multiplicity), and so requiring at least three multiple zeroes is a condition of codimension 3. Thus, we have

$$\dim X = \dim \mathbb{P}(V) - 3,$$

and as fibers of Z over $\mathbb{P}^{2,\vee}$ are isomorphic (as all lines in $\mathbb{P}^2$ are symmetric) the fiber dimension theorem tells us that

$$\dim Z = \dim(X) + 2 = \dim \mathbb{P}(V) - 1 < \dim \mathbb{P}(V),$$

so the map $Z \to \mathbb{P}(V)$ cannot be dominant, and so a generic curve has no tritangents.

5. (2 points, easier after Tuesday, extra credit) Let $\mathrm{Gr}(2,4)$ be the Grassmannian of two-dimensional subspaces in a four dimensional vector space V_4. (Equivalently, it parametrizes lines in $\mathbb{P}^3$, if you would prefer to think in those terms.)

   Choose fixed subspaces V_1, V_2, and V_3 of V_4 such that V_i has dimension i and $V_1 \subset V_2 \subset V_3$. For any nonnegative integers a_1, a_2, and a_3, let $X_{a_1,a_2,a_3} \subset \mathrm{Gr}(2,4)$ be the locus of two-dimensional subspaces W such that for each i, $W \cap V_i$ has dimension a_i. Show that each X_{a_1,a_2,a_3} is either empty or isomorphic to an affine space. (Don't worry too much about rigorously showing the isomorphism at the level of varieties - if you can prove that some algebraic map induces a bijection of sets that's good enough.) Examine the decomposition into affine spaces given by the X_{a_1,a_2,a_3} and explain why it suggests that $\mathrm{Gr}(2,4)$ is not isomorphic to a projective space.

Solution: The chain $V_1 \subset V_2 \subset V_3$ can be extended to the chain $V_0 \subset V_1 \subset V_2 \subset V_3 \subset V_4$, where V_0 contains just 0. Set $a_0 = 0$ and $a_4 = 2$ (as these must be the dimensions of the intersection of W with V_0 and V_2, respectively.)

Assume that X_{a_1,a_2,a_3} is nonempty. Then, in terms of the a_i, this implies that $a_0 = 0 \le a_1 \le a_2 \le a_3 \le a_4 = 2$, as each V_i is contained in V_{i+1}. It also implies that there is at most a gap of 1 between any two consecutive terms in the above chain of inequalities (as each V_i is of codimension 1 in V_{i+1}). Conversely, we claim that if the a_i satisfy these conditions, then the X_{a_1,a_2,a_3} are affine spaces.

Let m_1 be the smallest integer such that $a_{m_1} = 1$, and let m_2 be the smallest integer such that $a_{m_2} = 2$. Then we claim that the choice of W is equivalent to the choices of a 1-dimensional subspace l_1 of V_{m_1} which does not lie in V_{m_1-1} and a 1-dimensional subspace l_2 of V_{m_2}/l_1 which does not lie in V_{m_2-1}/l_1. Indeed, given such a W, we can set $l_1 = W \cap V_{m_1}$ and $l_2 = (W/l_1) \cap (V_{m_2}/l_1)$. Conversely, given l_1 and l_2, we can set W to be the preimage of l_2 under the map $V_{m_2} \to V_{m_2}/l_1$.

The choice of a 1-dimensional subspace l_1 of V_{m_1} which does not lie in V_{m_1-1} is equivalent to choosing a point in $\mathbb{P}(V_{m_1}) \cong \mathbb{P}^{m_1-1}$ which does not lie in $\mathbb{P}^{m_1-2}$, and the space of such choices is isomorphic to $\mathbb{A}^{m_1-1}$. Similarly, the space of choices of a l_2 is isomorphic to $\mathbb{A}^{m_2-2}$. Thus, the space of choices of W is isomorphic to the affine space $\mathbb{A}^{m_1+m_2-3}$, as desired.

Examining the numerics, we see that there are six nonempty cells X_{a_1,a_2,a_3}, namely:

(a) $X_{0,0,1} \cong \mathbb{A}^4$,
(b) $X_{0,1,1} \cong \mathbb{A}^3$,
(c) $X_{1,1,1} \cong \mathbb{A}^2$,
(d) $X_{0,1,2} \cong \mathbb{A}^2$,
(e) $X_{1,1,2} \cong \mathbb{A}^1$, and

(f) $X_{1,2,2} \cong \mathbb{A}^0$.

This suggests that $\mathrm{Gr}(2,4)$ is not a projective space, because our cell decomposition of $\mathrm{Gr}(2,4)$ has two $\mathbb{A}^2$s vs one for our cell decomposition of $\mathbb{P}^4$. (This is not a rigorous proof - however, if you know some algebraic topology, you can use the above to show that the 4th betti number of $\mathrm{Gr}(2,4)$ is 2 vs 1 for $\mathbb{P}^4$.)

6. (1 point) Choose a theorem from this class whose proof you don't fully understand. Try to explain the proof to somebody else (anybody you want). What theorem did you choose, and what did you learn from the process of explanation?

Solution: Many possible answers.