18.721 Final Project Topic Suggestions

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1 Introduction

This is a list of books/papers that could serve as inspiration for choosing a topic for your final presentation. For most of the references I have provided links - if you are on MIT SECURE, you should have institutional access to these references. If you are not, you should still be able to access them via the MIT Libraries proxy.

Most of these references are too extensive to form reasonable final project topics in their entirety, but will have many ideas to explore. I suggest looking for some individual aspect that seems interesting to you, and trying to understand just that one aspect. You may find your choice of topic changes as your understanding evolves— that's OK! Your final presentation should communicate a kernel of insight and not just be a cursory list of theorems and proofs.

The subjects described here form only a small portion of the full breadth of algebraic geometry. In particular, nothing on the applied side is mentioned here, largely because I do not know enough to provide good references. If you are interested in a subject not described here, email me, and I will probably be OK with it.

2 Potential References

2.1 Geometry of curves

- **David Mumford, Curves and their Jacobians**

  Gives a general summary of the theory of projective curves/Riemann surfaces, from both an algebraic and a differential-geometric point of view. Notably, it talks about the Jacobian of a curve, an abelian variety (that is, a projective variety which is also an abelian group) that one can attach to any curve.

- **Arbarello-Cornalba-Griffiths-Harris, Geometry of Algebraic Curves, Volume I**

  The standard reference book for the geometry of curves. Chapter I has a lot of overlap with the tail end of this class. Chapter III is the natural next step, proving some constraints on the triples (d, g, r) where there is an degree d embedding of a smooth genus g curve into $\mathbb{P}^r$. Also of note is Chapter V, which describes the complete solution to this problem when the curve is assumed to be generic.

- **Benson Farb and Dan Margalit, A Primer on Mapping Class Groups**

  A treatment on various aspects of the differential and hyperbolic geometry of Riemann surfaces/projective curves, from the perspective of the mapping class group. Contains a lot of fascinating material if you're interested in approach the subject from a hyperbolic geometry perspective.

2.2 Geometry of surfaces

- **Arnaud Beauville,Complex Algebraic Surfaces**

  The standard text on the Enriques-Kodaira classification of algebraic surfaces (smooth projective varieties of dimension 2). Good if you want a sense of the complexities of higher-dimensional algebraic geometry.

- **Miles Reid, Chapters on algebraic surfaces**

  An idiosyncratic treatment of many of the same topics from a different perspective.

2.3 Moduli spaces

- **Joe Harris and Ian Morrison, Moduli of Curves**

  It is an incredible theorem that for every nonnegative integer g, there is a natural algebraic variety, the moduli space of genus g curves, whose points correspond to isomorphism classes of smooth projective curves of genus g. The geometry of moduli of curves has been intensely studied from many perspectives. Unfortunately, it is very difficult to do moduli theory purely from a variety-theoretic perspective, and so this text is written in the language of schemes. It would be pretty difficult to understand knowing only the material in this class, but if anybody is really interested in moduli theory and would like to try to get something out of this book, I would recommend focusing on Chapter I, which constructs the parameter space of sub-schemes of projective space.

2.4 Number Theory

- **Joe Silverman, Arithmetic of Elliptic Curves**

  The standard first course in arithmetic geometry. The book addresses the question: When does an elliptic curve have points defined over $\mathbb{Q}$ (or $\mathbb{Z}$, or $\mathbb{F}_p$, or...)? Elliptic curves are the simplest nontrivial algebraic variety for which one can ask this question, and the (still partly conjectural) answer in this case is already very nontrivial.

- **Joe Silverman, Diophantine Geometry**

  If you are interested in understanding the arithmetic of varieties more complicated than elliptic curves, this is your book. It will be pretty hard to understand, but if you find it inspiring, you should be able to get something out of it.

- **Neal Koblitz, p-adic Numbers, p-adic Analysis, and Zeta-Functions**

  At first glance this doesn't have much to do with algebraic geometry, but the book actually builds up to Dwork's proof of some of the Weil Conjectures. (Which describe surprising patterns in the numbers of $\mathbb{F}_{p^r}$-points on an algebraic variety.)

2.5 Enumerative Geometry

- **Joe Harris, 3264 and all that**

  A second course in algebraic geometry built around trying to count objects. The title is a reference to the fact that there are 3264 conics in the plane tangent to 5 generic conics.

- **Ravi Vakil,The enumerative geometry of rational and elliptic curves in projective space**

  Counts curves satisfying certain conditions and has lots of nice pictures. As an example, see Figure 1 for an explanation of why there are $80,160$ cubics in $\mathbb{P}^3$ touching 12 general lines.

- **Joachim Kock and Israel Vainsencher, Kontsevich's Formula for Rational Plane Curves**

  Let R_d be the number of degree d maps $\mathbb{P}^1 \to \mathbb{P}^2$ through $3d-1$ general points (where two maps are considered the same if their images agree). We have $R_1 = 1$ (there is exactly one line through any 2 points) and $R_2 = 1$ (there is exactly one conic through any 5 points). Is there a general formula for R_d? Kontsevich proved the following recurrence relation:

  $$R_d = \sum_{\substack{d_A+d_B=d \\ d_A,d_B\geq 1}} \binom{3d-4}{3d_A-2} R_{d_A} R_{d_B} d_A^2 d_B^2 - \binom{3d-4}{3d_A-1} R_{d_A} R_{d_B} d_A^3 d_B.$$

  Interestingly, this formula comes from a relation between algebraic geometry and string theory, though no physics is needed to understand the proof.

2.6 Toric/tropical geometry

- **Cox-Little-Schenck, Toric Varieties**

  I didn't put a hyperlink, but you should be able to find a pdf as one of the top hits on google. As you might expect, this book goes more into depth on toric varieties, which are a great simple case where you can see how all the concepts of algebraic geometry work explicitly.

- **Diane Maclagan and Bernd Sturmfels, Introduction to tropical geometry**

  Another version of algebraic geometry where varieties are more combinatorial in nature and simpler to work with - google pictures of tropical curves and marvel at how simple they are.

2.7 Classic papers (in French)

These papers are all great, but they are also in French. If you speak French well, or if you are trying to improve your mathematical French (which is much easier than normal French), these could be fun to look into.

- **Jean-Pierre Serre, Faisceaux algébriques cohérents**

  The original paper on coherent sheaves in algebraic geometry.

- **Jean-Pierre Serre, Géométrie algébrique et géométrie analytique**

  Shows that various algebraic-geometric and complex-analytic notions sometimes agree. (As an example, though this was known before this paper: Any complex-analytic subvariety of projective space is in fact an algebraic subvariety.)

- **Alexander Grothendieck, Sur quelques points d'algèbre homologique**

  Sets up the general machinery of derived functors, of which sheaf cohomology is but one example.