18.721 PSet 7

Due: Apr 7, 11:59 PM

At the top of your submission, list all the sources you consulted, or write "**Sources consulted:** none" if you did not consult any sources.

1. (1 point) Recall that if R is a $\mathbb{C}$-algebra such that $\operatorname{Spec} R$ is a smooth curve, then for any maximal ideal m of R, the localization R_m of R by the complement of m is a DVR. Show that the same is true if R is the ring $\mathbb{Z}$ of integers. (This is part of a general philosophy that the integers behave like the ring of functions on a curve.)

2. (2 points) Recall the ring $\mathbb{C}[[x, y]]$ of formal power series in two variables that we introduced in PSet 3. Show that $\mathbb{C}[[x, y]]$ is a local ring. (The exercises/solutions for PSet 3 may be helpful.)

3. This week in class, we will prove Chevalley's theorem, as well as the related fact that for any map of varieties $f : X \to Y$ with X projective, the image of f is a closed subvariety of f. Let's demonstrate the power of this theorem by proving a version of semi-continuity of fiber dimension. We'll do this through a series of exercises, each one using the previous.

   (a) (1 point) Show that a connected projective variety over $\mathbb{C}$ has no non-constant maps to $\mathbb{A}^1$. (Hint: Use that a projective variety is compact in the classical topology.)

   (b) (2 points) Let $X \subseteq \mathbb{P}^n$ be a projective variety of dimension > 0. Show that every hyperplane in $\mathbb{P}^n$ intersects X. Furthermore, if X has dimension d, show that every $n - d$-dimensional linear space in $\mathbb{P}^n$ intersects X.

   (c) (2 points) Let Y be a variety and S be a closed subvariety of $Y \times \mathbb{P}^n$. There is a natural projection $f : S \to Y$. Show that for any integer d, the locus of points p in Y where the fiber of f above p is nonempty and has dimension $\geq d$ forms a closed subvariety of Y.

   (d) (1 point) Let $f : X \to Y$ be a map of varieties with X projective and let d be an integer. Again, show that the locus of points p in Y where the fiber of f above p is nonempty and has dimension $\geq d$ forms a closed subvariety of Y. (The notion of the graph of a morphism may come in handy.)

(e) (1 point) Show that if X is not projective, the conclusion of the above exercise may not necessarily be true. (What is true in general is that the locus of points q in X (not Y) where q is contained in a component of $f^{-1}(f(q))$ of dimension at least d is closed in X. When X is projective, the image of this closed set will be a closed set of Y, and we recover the conclusion of the above exercises. We will not prove this more general statement of semicontinuity of fiber dimension.)

4. (1 point) Look at the list of potential references for final projects posted on Canvas. See what interests you and choose a potential topic for your final project. (This choice is not binding in any way.)