18.721 PSet 8

Due: Apr 12, 11:59 PM

At the top of your submission, list all the sources you consulted, or write "**Sources consulted:** none" if you did not consult any sources.

1. Let $\mathbb{R}$ be the real numbers, equipped with its standard topology. We'll consider some presheaves on $\mathbb{R}$.
   (a) (1 point) Let $\mathcal{F}$ be the presheaf where $\mathcal{F}(U)$ is the space of continuous real-valued functions on U (and the restriction maps are given by restriction of functions). Show that $\mathcal{F}$ is a sheaf.
   (b) (1 point) Let $\mathcal{G}$ be the presheaf where $\mathcal{G}(U)$ is the space of constant functions on U. Show that $\mathcal{G}$ is NOT a sheaf.
2. (2 points, Artin Exercise 6.9.13) Let $\mathcal{F}$ be a quasicoherent sheaf on X, and let s be a nonzero regular function defined on all of X. Show that the localization $\mathcal{F}(X)_s$ is isomorphic to $\mathcal{F}(X_s)$. (This follows from the definition of a quasi-coherent sheaf when X is affine, but it is not immediate when X is not affine.)
3. (2 points) Consider the map $f : \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}$ corresponding to multiplication by xy, where we take (x, y) to be projective coordinates for $\mathbb{P}^1$. Let $\mathcal{H}$ be the presheaf such that $\mathcal{H}(U)$ can be identified with the cokernel of $f(U) : \mathcal{O}(-2)(U) \to \mathcal{O}(U)$. Show that $\mathcal{H}$ is not a sheaf, and describe how it differs from the sheaf cokernel of f.
4. For our construction of cokernels of sheaves, we took the naive cokernel (as in the previous problem) and then turned it into a sheaf using Artin's Theorem 6.3.2, which constructs a quasicoherent sheaf from its values on affine opens. There is a more general "sheafification" functor which we now describe, which produces a sheaf $\mathcal{F}^{\#}$ from a presheaf $\mathcal{F}$ on any topological space X.
   (a) (1 point) Let S_p be the set of pairs (U, f) where U is an open subset of X containing p and f is an element of $\mathcal{F}(U)$. Say that two such pairs (U, f) and (V, g) are equivalent if there is some open W such that:
      - W contains p,

- W is contained in both U and V, and
- the restrictions of f and g to W agree.

Show that this is an equivalence relation on S_p. The set of equivalence classes is called the stalk $\mathcal{F}_p$ of $\mathcal{F}$ at p.

(b) (1 point) Let X be $\operatorname{Spec} A$, with the Zariski topology, and let p correspond to a maximal ideal m. Let $\mathcal{F}$ be the quasi-coherent sheaf corresponding to a module M. Show that the stalk $\mathcal{O}_p$ is the local ring A_m, and that the stalk $\mathcal{F}_p$ is the localization M_m.

(c) (2 points) Returning to the case of a general topological space X, define $\mathcal{F}^\#$ by letting $\mathcal{F}^\#(U) \subseteq \prod \mathcal{F}_p$ be the set of collections of elements $f_p \in \mathcal{F}_p$, one f_p for each point p of U, where the collection satisfies the following property:

- there exists a (possibly infinite) cover of U by opens U_i and a collection of sections $f_i \in \mathcal{F}(U_i)$ such that for every point p of U_i, the equivalence class of (U, f_i) in $\mathcal{F}_p$ is f_p.

Show that $\mathcal{F}^\#$ is a sheaf.

(d) (1 point, extra credit) Let X be a variety and let $\mathcal{F}$ be a quasi-coherent presheaf on X. Show that $\mathcal{F}^\#(U)$ agrees with $\mathcal{F}(U)$ for any affine open U. Thus, this sheafification procedure is the same as the one used in Artin's Theorem 6.3.2.

5. (1 point) Look back at your answer to the last problem of Problem Set 4 (where you had to look up the definition of a sheaf.) How has your thinking around sheaves changed now that we've started formally working with them?