18.721 PSet 7

Due: Apr 7, 11:59 PM

At the top of your submission, list all the sources you consulted, or write "**Sources consulted:** none" if you did not consult any sources.

1. (1 point) Recall that if R is a $\mathbb{C}$-algebra such that $\operatorname{Spec} R$ is a smooth curve, then for any maximal ideal m of R, the localization R_m of R by the complement of m is a DVR. Show that the same is true if R is the ring $\mathbb{Z}$ of integers. (This is part of a general philosophy that the integers behave like the ring of functions on a curve.)

Solution: The maximal ideals of $\mathbb{Z}$ are (p) for p a prime number. Fix a prime number p and let $m = (p)$. Then R_m is the ring of rational numbers of the form $\frac{a}{b}$, where a and b are integers and b is not divisible by p.

We define a valuation v_p on $\mathbb{Q}$ as follows. For any rational number r, write it as $\frac{a}{b}$, and assume that the largest power of p dividing a is p^i and that the largest power of p dividing b is p^j. Then we set $v_p(r) = i - j$. It is straightforward to check that this defines a valuation. The set of rational numbers with $v_p(r) \geq 0$ are those with $j \leq i$, which means that we have

$$\frac{a}{b} = \frac{\frac{a}{p^j}}{\frac{b}{p^j}},$$

and both the numerator and the denominator are integers. Since p^j is the maximal power of p that divides b, the denominator is not a multiple of p, so $\frac{a}{b}$ is an element of $\mathbb{Z}_m$. Conversely, every element r of $\mathbb{Z}_m$ has $v_p(r) \geq 0$, so $\mathbb{Z}_m$ is a DVR, as desired.

2. (2 points) Recall the ring $\mathbb{C}[[x, y]]$ of formal power series in two variables that we introduced in PSet 3. Show that $\mathbb{C}[[x, y]]$ is a local ring. (The exercises/solutions for PSet 3 may be helpful.)

Solution: A maximal ideal m of $\mathbb{C}[[x, y]]$ is given by (x, y). We will show that every element not in m is invertible, which is equivalent to $\mathbb{C}[[x, y]]$ being a local ring with m as its maximal ideal.

This can be done in a way along the lines of parts 4b) and 4c) of PSet 3. We will do this instead with a trick. Let f be an element of $\mathbb{C}[[x,y]]$ which is not in m. It can be written as $c(1+g)$, where c is a nonzero complex number and g is an element of m. It suffices to show that $1+g$ is invertible. An explicit inverse is given by

$$1 - g + g^2 - g^3 +, \cdots$$

where this series converges because the g^i are elements of m^i, and so we can apply Problem 4a) from PSet 3.

3. This week in class, we will prove Chevalley's theorem, as well as the related fact that for any map of varieties $f : X \to Y$ with X projective, the image of f is a closed subvariety of f. Let's demonstrate the power of this theorem by proving a version of semi-continuity of fiber dimension. We'll do this through a series of exercises, each one using the previous.

   (a) (1 point) Show that a connected projective variety over $\mathbb{C}$ has no non-constant maps to $\mathbb{A}^1$. (Hint: Use that a projective variety is compact in the classical topology.)

Solution: Let f be a map from a connected projective X to $\mathbb{A}^1$. Let i be the standard open embedding of $\mathbb{A}^1$ into $\mathbb{P}^1$. The image of $i \circ f$ must be a Zariski closed subset of $\mathbb{P}^1$, but must also be contained in the open set $\mathbb{A}^1$. This implies that the image is a finite set of points. Since X is connected, the image must be just one point, so f is constant.

   (b) (2 points) Let $X \subseteq \mathbb{P}^n$ be a projective variety of dimension > 0. Show that every hyperplane in $\mathbb{P}^n$ intersects X. Furthermore, if X has dimension d, show that every $n-d$-dimensional linear space in $\mathbb{P}^n$ intersects X.

Solution: We can assume X is irreducible (if not, replace X by an irreducible component which still has positive dimension). Assume that some hyperplane H does not intersect X. We can choose coordinates so that H is the complement of the standard $\mathbb{A}^n$ inside $\mathbb{P}^n$. As X does not intersect H, this implies that X is a subvariety of $\mathbb{A}^n$. But by part (a), the projection of X to any part $\mathbb{A}^1$ must be constant, so X must be just one point (and hence zero-dimensional), a contradiction.

For the second part, we use induction on d. Again, we can assume that X is irreducible. The first part gives the case $d = 1$. Assume $d > 1$, and let L be the $n-d$-dimensional linear space. It must be contained in some hyperplane H. The intersection $H \cap X$ is nonempty by the first part. Choose a point p in the intersection, and choose some affine open isomorphic to $\mathbb{A}^n$ containing p. Then $H \cap \mathbb{A}^n$ is the vanishing locus of one function in $\mathbb{A}^n$,, so $H \cap X \cap \mathbb{A}^n$

must be of dimension at least $d-1$, and so $H \cap X$ itself must be of dimension at least $d-1$. The inductive hypothesis for $H \cap X$ inside $H \cong \mathbb{P}^{n-1}$ then tells us that $H \cap X$ intersects L, so X must intersect L as desired.

The above logic can also be used to conclude that if X has dimension less than d, then some $n-d$-dimensional linear space does not intersect X. If $d=1$, then this is clear. In general, we use the same induction, and replace X with its intersection with a generic hyperplane H, and then proceed as in the above proof. The only new fact we need to show that is $X \cap H$ has dimension exactly $d-1$ (and not d). Krull's theorem tells us that this will hold as long as H does not contain (an irreducible component of) X, which is easy to arrange.

(c) (2 points) Let Y be a variety and S be a closed subvariety of $Y \times \mathbb{P}^n$. There is a natural projection $f : S \to Y$. Show that for any integer d, the locus of points p in Y where the fiber of f above p is nonempty and has dimension $\geq d$ forms a closed subvariety of Y.

Solution: In our solution to the previous part, we showed that the desired locus is the same as the locus of points p where the fiber intersects L for every $n-d$-dimensional linear space L. For a fixed L, call the locus where this intersection is nonempty S_L. This is the same as the image of $S \cap (Y \times L)$ inside Y, which is closed as $L \cong \mathbb{P}^{n-d}$ is proper. The desired locus is the intersection of the S_L as L ranges over all $n-d$-dimensional linear spaces, and thus is closed again, as desired.

(d) (1 point) Let $f : X \to Y$ be a map of varieties with X projective and let d be an integer. Again, show that the locus of points p in Y where the fiber of f above p is nonempty and has dimension $\geq d$ forms a closed subvariety of Y. (The notion of the graph of a morphism may come in handy.)

Solution: Let $Z \subseteq X \times Y \subseteq \mathbb{P}^n \times Y$ be the graph of f. The locus we are studying in this part is the same as the locus of points p in Y where the fiber of $Z \to Y$ is nonempty of dimension $\geq d$, so we can apply the result of the previous part.

(e) (1 point) Show that if X is not projective, the conclusion of the above exercise may not necessarily be true. (What is true in general is that the locus of points q in X (not Y) where q is contained in a component of $f^{-1}(f(q))$ of dimension at least d is closed in X. When X is projective, the image of this closed set will be a closed set of Y, and we recover the conclusion of the above exercises. We will not prove this more general statement of semicontinuity of fiber dimension.)

Solution: As a simple example, we can take $X \to Y$ to be the open immersion $\mathbb{A}^1 \to \mathbb{P}^1$. Then the locus of points q in Y where the fiber is nonempty (hence nonempty of dimension at least 0) is $\mathbb{A}^1$, which is not closed in $\mathbb{P}^1$.

4. (1 point) Look at the list of potential references for final projects posted on Canvas. See what interests you and choose a potential topic for your final project. (This choice is not binding in any way.)

Solution: Many possible solutions.