18.721 PSet 8

Due: Apr 12, 11:59 PM

At the top of your submission, list all the sources you consulted, or write "**Sources consulted:** none" if you did not consult any sources.

1. Let $\mathbb{R}$ be the real numbers, equipped with its standard topology. We'll consider some presheaves on $\mathbb{R}$.

   (a) (1 point) Let $\mathcal{F}$ be the presheaf where $\mathcal{F}(U)$ is the space of continuous real-valued functions on U (and the restriction maps are given by restriction of functions). Show that $\mathcal{F}$ is a sheaf.

Solution: Assume we have an open cover $\{U_i\}$ of an open set U. Let $\{f_i \in \mathcal{F}(U_i)\}$ be a collection of sections such that for every pair (i, j), the restrictions of f_i and f_j to $U_i \cap U_j$ agree. Then for every point p, the value of $f_i(p)$ is the same for every i such that $p \in U_i$. Indeed, if there are two different i, j such that p is in both U_i and U_j, then p is in $U_i \cap U_j$, so the values of f_i and f_j at p agree.

Let f be the function such that $f(p)$ is equal to the value of any of the $f_i(p)$ for i such that U_i contains p. We claim that is f is continuous, and hence corresponds to a global section of $\mathcal{F}$. Indeed, for any point p, choose a U_i containing p. Then f coincides with f_i on U_i, so f must be continuous at p. This gives a global section of $\mathcal{F}$ whose restriction to each U_i is the f_i.

Conversely, if g were a different global section which restricted to each of the U_i, we must have $g(p) = f_i(p) = f(p)$ for any point p and any open U_i containing p. Thus, g would have to coincide with f, so we have shown both the existence and uniqueness of the gluing of the $\{f_i\}$.

   (b) (1 point) Let $\mathcal{G}$ be the presheaf where $\mathcal{G}(U)$ is the space of constant functions on U. Show that $\mathcal{G}$ is NOT a sheaf.

Solution: Let U_1 be the open interval $(0, 1)$ and let U_2 be the open interval $(2, 3)$. As $U_1 \cap U_2$ is empty, the sheaf property for the open cover $\{U_1, U_2\}$ of $U_1 \cup U_2$ states just that the natural map $\mathcal{F}(U_1 \cup U_2) \to \mathcal{F}(U_1) \times \mathcal{F}(U_2)$. But in this case, this map is the map

$$\mathbb{R} \to \mathbb{R} \times \mathbb{R}$$

sending r to (r, r), which is clearly not an isomorphism.

2. (2 points, Artin Exercise 6.9.13) Let $\mathcal{F}$ be a quasicoherent sheaf on X, and let s be a nonzero regular function defined on all of X. Show that the localization $\mathcal{F}(X)_s$ is isomorphic to $\mathcal{F}(X_s)$. (This follows from the definition of a quasi-coherent sheaf when X is affine, but it is not immediate when X is not affine.)

Solution: The sheaf property gives us the exact sequence

$$0 \to \mathcal{F}(X) \to \prod \mathcal{F}(U_i) \to \prod \mathcal{F}(U_i \cap U_j)$$

for any cover of X by opens U_i. Choose the U_i to be affine. As localization is exact, we get an exact sequence

$$0 \to \mathcal{F}(X)_s \to \prod \mathcal{F}(U_i)_s \to \prod \mathcal{F}(U_i \cap U_j)_s.$$

By quasi-coherence of $\mathcal{F}$, we have isomorphisms $\mathcal{F}(U_i)_s \cong \mathcal{F}((U_i)_s)$. As the intersection of two affines is affine, we also have isomorphisms $\mathcal{F}(U_i \cap U_j)_s \cong \mathcal{F}((U_i)_s \cap (U_j)_s)$. Together, these show that the above exact sequence is isomorphic to

$$0 \to \mathcal{F}(X)_s \to \prod \mathcal{F}((U_i)_s) \to \prod \mathcal{F}((U_i)_s \cap (U_j)_s),$$

so $\mathcal{F}(X)_s$ is the kernel of the map

$$\prod \mathcal{F}((U_i)_s) \to \prod \mathcal{F}((U_i)_s \cap (U_j)_s).$$

The sheaf property for the open cover of X_s by the $(U_i)_s$ tells us this kernel is isomorphic to $\mathcal{F}(X_s)$, as desired.

3. (2 points) Consider the map $f : \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}$ corresponding to multiplication by xy, where we take (x, y) to be projective coordinates for $\mathbb{P}^1$. Let $\mathcal{H}$ be the presheaf such that $\mathcal{H}(U)$ can be identified with the cokernel of $f(U) : \mathcal{O}(-2)(U) \to \mathcal{O}(U)$. Show that $\mathcal{H}$ is not a sheaf, and describe how it differs from the sheaf cokernel of f.

Solution: Let $U_0 \subset \mathbb{P}^1$ be the complement of $(0, 1)$ and let $U_1 \subset \mathbb{P}^1$ be the complement of $(1, 0)$. We can identify the restriction of $\mathcal{O}(n)$ to U_0 with the vector space of ratios $\frac{P(x,y)}{x^i}$ where P is homogeneous of degree $i + n$. Then the image of the restriction of f to U_0 consists of such ratios $\frac{P(x,y)}{x^i}$ where P is homogeneous of degree i and a multiple of y. The quotient of $\mathcal{O}(U_0)$ by this image is a 1-dimensional vector space spanned by the image of 1, so $\mathcal{H}(U_0)$ is 1-dimensional. Similarly, $\mathcal{H}(U_1)$ is 1-dimensional.

On the other hand, xy is invertible on $U_0 \cap U_1$, so $\mathcal{H}(U_0 \cap U_1)$ is the trivial vector space. The global sections of $\mathcal{H}$ can easily be seen to correspond to the 1-dimensional vector space of constant functions. Then the sheaf property for the open cover of $\mathbb{P}^1$ by U_0 and U_1 would then say that

$$0 \to \mathbb{C} \to \mathbb{C}^2 \to 0$$

is exact, which is clearly false. The sheaf $\mathcal{H}^\sharp$ can only differ from $\mathcal{H}$ on non-affine opens, so in the above exact sequence, the only term that could differ would be $\mathcal{H}(\mathbb{P}^1)$, which would have to be 2-dimensional.

4. For our construction of cokernels of sheaves, we took the naive cokernel (as in the previous problem) and then turned it into a sheaf using Artin's Theorem 6.3.2, which constructs a quasicoherent sheaf from its values on affine opens. There is a more general "sheafification" functor which we now describe, which produces a sheaf $\mathcal{F}^\#$ from a presheaf $\mathcal{F}$ on any topological space X.

   (a) (1 point) Let S_p be the set of pairs (U, f) where U is an open subset of X containing p and f is an element of $\mathcal{F}(U)$. Say that two such pairs (U, f) and (V, g) are equivalent if there is some open W such that:

   - W contains p,
   - W is contained in both U and V, and
   - the restrictions of f and g to W agree.

   Show that this is an equivalence relation on S_p. The set of equivalence classes is called the stalk $\mathcal{F}_p$ of $\mathcal{F}$ at p.

Solution: Reflexivity and symmetry are immediate. The only remaining property to check is transitivity. Assume (U_1, f_1) is equivalent to (U_2, f_2) and (U_2, f_2) is equivalent to (U_3, f_3). Then there must be open neighborhoods $W_1 \subseteq U_1 \cap U_2$ and $W_2 \subseteq U_2 \cap U_3$ of p such that the restrictions of f_1 and f_2 to W_1 agree and the restrictions of f_2 and f_3 to W_2 agree. Then let $W = W_1 \cap W_2$. It is clearly an open neighborhood of p, and we have $W \subseteq W_1 \subseteq U_1$ and $W \subseteq W_2 \subseteq U_3$.
Since $W \subseteq W_1$, the restrictions of f_1 and f_2 to W agree. Similarly, the restrictions of f_2 and f_3 to W agree. It follows that the restrictions of f_1 and f_3 to W agree, so (U_1, f_1) is equivalent to (U_3, f_3).

   (b) (1 point) Let X be $\operatorname{Spec} A$, with the Zariski topology, and let p correspond to a maximal ideal m. Let $\mathcal{F}$ be the quasi-coherent sheaf corresponding to a module M. Show that the stalk $\mathcal{O}_p$ is the local ring A_m, and that the stalk $\mathcal{F}_p$ is the localization M_m.

Solution: We show this for $\mathcal{F}_p$. The case of $\mathcal{O}_p$ then follows by taking $\mathcal{F} \cong \mathcal{O}$ and checking that the ring structures on both sides agree.

We define a map from M_m to $\mathcal{F}_m$ as follows. Let $\frac{y}{x}$ be an element in M_m with x in $A \backslash m$ and $y \in M$. If we set U to be Spec A_y, then $\mathcal{F}(U) \cong \mathcal{F}(\text{Spec } A)_y \cong M_y$, so we can interpret $\frac{y}{x}$ as an element of $\mathcal{F}(U)$. Then the image of $\frac{y}{x}$ in $\mathcal{F}_m$ is defined to be the equivalence class of $(U, \frac{y}{x})$.

The inverse map is given as follows. For every equivalence as in part (a), choose a representative (U, f) where U is a distinguished affine Spec A_s. Then f can be written as a ratio $\frac{x}{s^i}$. The condition that p is in U tells us that s is not in m, and so $\frac{x}{s^i}$ can be interpreted as an element of A_m. It is straightforward to check that this element does not depend on the choice of representative, and that these two maps are inverses, as desired.

(c) (2 points) Returning to the case of a general topological space X, define $\mathcal{F}^{\#}$ by letting $\mathcal{F}^{\#}(U) \subseteq \prod \mathcal{F}_p$ be the set of collections of elements $f_p \in \mathcal{F}_p$, one f_p for each point p of U, where the collection satisfies the following property:

- there exists a (possibly infinite) cover of U by opens U_i and a collection of sections $f_i \in \mathcal{F}(U_i)$ such that for every point p of U_i, the equivalence class of (U, f_i) in $\mathcal{F}_p$ is f_p.

Show that $\mathcal{F}^{\#}$ is a sheaf.

Solution: Let U_i be an open cover of an open subset U, and let f_i be elements of $\mathcal{F}(U_i)$ which agree on the intersections $U_i \cap U_j$. Each f_i corresponds to a collection $f_{i,p}$, where p ranges over all points in U_i.

For any point p, all the $f_{i,p} \in \mathcal{F}_p$ which are defined must agree. (This follows from the same logic as in Part (a) of Problem 1). We can then define a global section f of $\mathcal{F}$ with f_p equal to the common value of all the $f_{i,p}$. The property in the problem statement must still hold for $\mathcal{F}$, by taking the cover of U to be the union of the open covers of all the U_i, so f indeed defines a global section which glues together all the f_i. It is easy to see (again, with the same logic as in Part (a) of Problem 1) that any other global section gluing together the f_i must be equal to f, so we have proven the sheaf property.

(d) (1 point, extra credit) Let X be a variety and let $\mathcal{F}$ be a quasi-coherent presheaf on X. Show that $\mathcal{F}^{\#}(U)$ agrees with $\mathcal{F}(U)$ for any affine open U. Thus, this sheafification procedure is the same as the one used in Artin's Theorem 6.3.2.

Solution: Let the quasicoherent sheaf constructed via Artin's Theorem 6.3.2 be denoted by $\mathcal{F}^{\flat}$. We will construct injections $\mathcal{F}(U) \to \mathcal{F}^{\#}(U) \to \mathcal{F}^{\flat}(U)$. Artin's theorem tells us that the composition of these two injections is an isomorphism, so in fact both injections must themselves be isomorphisms, as desired.

The map $\mathcal{F}(U) \to \mathcal{F}^{\#}(U)$ is defined by sending an element x to the collection of the images of (U, x) in each $\mathcal{F}_p$. It remains to show that this is injective. Assume there is some element x which maps to 0 in each $\mathcal{F}_p$. Then that means that for each p, there is some open set U_p around p for which the restriction of x to U_p is 0. Then the sheaf property with respect to the open cover given by the U_p tells us that x must be zero, as desired.

It remains to define the map $\mathcal{F}^{\#}(U) \to \mathcal{F}^{\flat}(U)$. Assume we have a collection $\{f_p\}$ satisfying the property in Part (c) for some open cover $\{U_i\}$ and some choice of sections $f_i \in \mathcal{F}(U_i)$. The restrictions of f_i and f_j to $U_i \cap U_j$ correspond to the same element f_p at every point $p \in U_i \cap U_j$, or in the other words, f_i and f_j have the same image in $\mathcal{F}^{\flat}(U_i \cap U_j)$. By the injectivity proven in the previous paragraph, this implies that they have the same image in $\mathcal{F}(U_i \cap U_j)$. But then the sheaf property tells us that they must glue to some element $f \in \mathcal{F}^{\flat}(U)$, which gives us the desired map $\mathcal{F}^{\#}(U) \to \mathcal{F}^{\flat}(U)$. To see injectivity, note that the sheaf property would imply that any element sent to 0 would have to have the f_i equal to 0, so the starting element of $\mathcal{F}^{\#}(U)$ would also have to be zero.

5. (1 point) Look back at your answer to the last problem of Problem Set 4 (where you had to look up the definition of a sheaf.) How has your thinking around sheaves changed now that we've started formally working with them?

Solution: Many possible solutions.