18.721 PSet 9

Due: Apr 19, 11:59 PM

At the top of your submission, list all the sources you consulted, or write "**Sources consulted:** none" if you did not consult any sources.

1. (2 points) Let X be the complement of the origin in $\mathbb{A}^2$. Compute $H^i(\mathcal{O}_X)$, and use this to give another argument that X is not affine.

2. (2 points, Artin 7.10.3) Let
$$0 \to V_0 \to V_1 \to \cdots \to V_n \to 0$$
be a complex of finite-dimensional vector spaces, and let $C_i \cong \ker(V_i \to V_{i+1})/\operatorname{im}(V_{i-1} \to V_i)$ be the ith cohomology of this complex. Show that
$$\sum_{i=0}^{n}(-1)^i \dim V_i = \sum_{i=0}^{n}(-1)^i \dim C_i.$$

3. Let R be a commutative ring. We say that a R-module M is projective if there is an integer n and another R-module N such that there is an isomorphism of R-modules $M \oplus N \cong R^n$.

   (a) (2 points) If R is a finite type $\mathbb{C}$-algebra, show that the quasicoherent sheaf corresponding to a projective module M as above is a vector bundle on $\operatorname{Spec} R$. (Hint: For any point p in $\operatorname{Spec} R$ corresponding to a maximal ideal m, choose a basis of M/mM and lifts m_i of the basis elements to M. Show that M becomes isomorphic to the free module generated by these elements after localization by some element of R not vanishing at p. One possible way to show this is by choosing similar elements n_i of N, and then examining when the m_i and n_i collectively generate $M \oplus N \cong R^n$.)

   (b) (1 point) It is slightly difficult at the moment for us to exhibit a finite type $\mathbb{C}$-algebra with modules that are projective but not free. However, there are simple examples coming from number theory. Recall that the ring $R = \mathbb{Z}[\sqrt{-5}]$ is not a principal ideal domain because the ideal $(2, 1+\sqrt{-5})$ is not principal. Show that $(2, 1+\sqrt{-5})$ is projective but not free as a module.

4. Note that the definition of Čech cohomology still makes sense for general (i.e. not necessarily quasicoherent) sheaves of abelian groups on X (with the Zariski topology). More precisely, for such a sheaf $\mathcal{F}$, we can define the ith Čech cohomology $H^i(\mathcal{F}, \{U_i\})$ of $\mathcal{F}$ with respect to a cover by opens U_i as the ith cohomology of the Čech complex

$$\prod \mathcal{F}(U_i) \to \prod \mathcal{F}(U_i \cap U_j) \to \prod \mathcal{F}(U_i \cap U_j \cap U_k) \cdots .$$

   (a) (1 point) Let $\mathcal{O}^*$ be the sheaf with $\mathcal{O}^*(U)$ defined to be the multiplicative group of invertible elements of $\mathcal{O}(U)$. Assume we have an open cover of X by two affine opens U_1, U_2. Compute $H^1(\mathcal{O}^*, \{U_1, U_2\})$ explicitly.

   (b) (2 points) Use your description from the previous part to show that $H^1(\mathcal{O}^*, \{U_1, U_2\})$ is isomorphic to the group of line bundles on X which become trivial when restricted to both U_1 and U_2.

   You may use the following result, which is a generalization of Artin's proposition 6.4.7 (and has the same proof):

   Lemma 1. *Let S be the set of triples $\{\mathcal{L}, f_1 : \mathcal{L}|_{U_1} \to \mathcal{O}(U_1), f_2 : \mathcal{L}|_{U_2} \to \mathcal{O}(U_2)\}$ such that f_1 and f_2 are isomorphisms (and so in particular, $\mathcal{L}$ is a line bundle). Then elements of S are in bijection with isomorphisms $\mathcal{O}(U_1)|_{U_1 \cap U_2} \to \mathcal{O}(U_2)|_{U_1 \cap U_2}$. The bijection sends an element $\{\mathcal{L}, f_1, f_2\}$ to the isomorphism*

$$\mathcal{O}(U_1)|_{U_1 \cap U_2} \xrightarrow{f_1^{-1}} \mathcal{L}_{U_1 \cap U_2} \xrightarrow{f_2} \mathcal{O}(U_2)|_{U_1 \cap U_2}.$$

   In particular, if we knew there existed an affine variety X with such a line bundle (as suggested by the previous problem), this would show that the higher Čech cohomology groups do not always vanish on affine varieties if we are looking at non-quasicoherent sheaves.

   (The statement of this problem in fact still holds for an affine open cover with more than two opens, but is significantly conceptually harder.)

5. (1 point) Try to imagine how you might have come up with the concept of sheaf cohomology. Give as plausible an explanation as you can think of.