18.721 PSet 10

Due: Apr 28, 11:59 PM

At the top of your submission, list all the sources you consulted, or write "**Sources consulted:** none" if you did not consult any sources.

For this problem set, whenever we write "curve", we mean "irreducible projective curve."

1. (1 point, Artin Exercise 8.10.1) Show that if C is a smooth curve of genus 0, then C is isomorphic to $\mathbb{P}^1$.

2. (2 points) Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Show that $\chi(\mathcal{F}(n))$ is a polynomial in n, and that the degre of this polynomial is the equal to the dimension of the support of $\mathcal{F}$. When $\mathcal{F}$ is the sheaf $i_*\mathcal{O}_X$, for $i : X \to \mathbb{P}^n$ a closed embedding, this polynomial is called the Hilbert polynomial of X.

3. Let's apply the concept of a Hilbert polynomial to study some curves.

   (a) (1 point) Let $C \subseteq \mathbb{P}^n$ be a smooth curve of degree d and genus g. Show that the Hilbert polynomial of C is $P(n) = dn + 1 - g$.

   (b) (2 points) Calculate the degree and genus of a smooth transverse intersection of two surfaces in $\mathbb{P}^3$ of degrees d and e.

4. Earlier in the course, we showed that for every curve X, not necessarily smooth, there is a smooth curve Y birational to X (recall that two varieties are birational if they have isomorphic open subvarieties.) We did this using the algebraic notion of normalization. Now that we've studied sheaf cohomology, we can give a geometric proof.

   We defined divisors and their associated line bundles $\mathcal{O}(D)$ only for smooth curves. However, for a singular curve, one can still define a line bundle $\mathcal{O}(D)$ (in the same way) for any divisor D containing only smooth points.

   (a) (1 point) Let C be a (possibly singular) projective curve, and let p be a smooth point of C. Show that there is some constant c such that we have
   $$H^0(\mathcal{O}(np)) > n - c$$
   for all $n \geq 0$.

(b) (1 point) Using the previous part, show that there is an embedding of C as a nondegenerate (i.e., not contained in a hyperplane) degree d curve in some $\mathbb{P}^m$ with $d < 2m$.

(c) (1 point) Show that a nondegenerate curve in $\mathbb{P}^n$ must have degree at least n. (Hint: Use induction, and consider the projection away from a point of the curve.)

(d) (1 point) Show that a nondegenerate curve in $\mathbb{P}^n$ with degree less than $2n$ must be birational to a smooth curve. (Hint: Try to argue as in the previous part, but this time projecting away from a singular point.)

5. (1 point) What theorem from the class was your favorite, and why?